SONAM BANSAL

COMPLEX ANALYSIS

05/11/14

REAL ANALYSIS

UNIFORM CONVERGENT

ABEL'S TEST :→ $\sum u_n(x) \cdot v_n(x)$, $x \in S$

1. $v_n(x)$ is monotonic.
2. $\langle v_n(x) \rangle$ is uniformly bounded.
3. $\sum u_n(x)$ is U.C.

Then $\sum u_n(x) \cdot v_n(x)$ is U.C.

Exp. $\sum a_n x^n$, $x \in [0, \alpha]$, $\alpha < 1$

$\Leftarrow \sum a_n$ is convergent.

$\sum u_n = \sum a_n$

$\sum a_n x^n$ is U.C. (given)

$\langle x^n \rangle$ is uniformly bounded.

$\langle x^n \rangle$ Monotonic

$\sum a_n$ is U.C.

Result :- $\sum u_n(x)$, $S_n(x) = \sum_{r=1}^{n} u_r(x)$

define , $\lim_{n \to \infty} S_n(x) = S(x)$

Where $S(x)$ is the PWL of $\langle S_n \rangle$ if $\langle S_n \rangle$ is U.C. then $S(x)$ is define.

$$\sum_{n=1}^{\infty} u_n(x) = S(x)$$

$\Rightarrow$

NOTE :- $\sum u_n(x)$ U.C. $\rightarrow S(x)$

(i) If each u_n is continuous $\Rightarrow S$ is conti.

(ii) If $u_n \in R[a, b] \Rightarrow S \in R[a, b]$

$$\Rightarrow \int_a^b \sum u_n(x)\, dx = \sum \int_a^b u_n(x)\, dx$$

3. $\sum u_n(x) = S(x)$ (U.C.)
if $\sum u'_n(x) = T(x)$ (U.C.)
then $S'(x) = T(x)$

4. $\sum u_n(x) = S(x)$
where $\lim_{n\to\infty} S_n(x) = S(x)$ (PWL)

Then for any $\alpha \in [a, b]$
$S(\alpha)$ is the sum of series of real no. $\sum u_n(\alpha)$

The function $S(x)$ is defined as sum function of $\sum u_n(x)$
(Sum of diff & conti. fun^c is diff. & conti.)

Exp. $f_n(x) = x\left(1 - \frac{1}{n}\right)$, $x \in (0, 1]$

$g_n(x) = \frac{1}{x^2}$, $x \in (0, 1]$

$f_n(x)$ & $g_n(x)$ are U.C.

If f + g are U.C. for any α $\alpha f + \beta g$ will be U.C. But Product will not U.C.

$h_n(x) = f_n(x) \cdot g_n(x)$
$= \frac{1}{x}\left(1 - \frac{1}{n}\right)$

$\lim_{n\to\infty} h_n(x) = \frac{1}{x}$

$|h_n(x) - h(x)| = \left|\frac{1}{x} - \frac{1}{nx} - \frac{1}{x}\right|$

$y = \frac{1}{nx}$

$y' = \frac{-1}{nx^2} = 0 \neq 0$

$\therefore$ not U.C.

#:> $f: \mathbb{R} \to \mathbb{R}$

$f'(x)$ is uniformly convergent. continuous.

Define,

$$g_n(x) = n\left(f(x+1/n) - f(x)\right)$$

$\langle g_n(x) \rangle$ is U.C.

$\lim g_n(x) = f'(x)$

$\langle g_n(x) \rangle \xrightarrow{U.C.} f'(x)$

$$|g_n(x) - f'(x)| = \left| \frac{f(x+1/n) - f(x)}{1/n} - f'(x) \right|$$

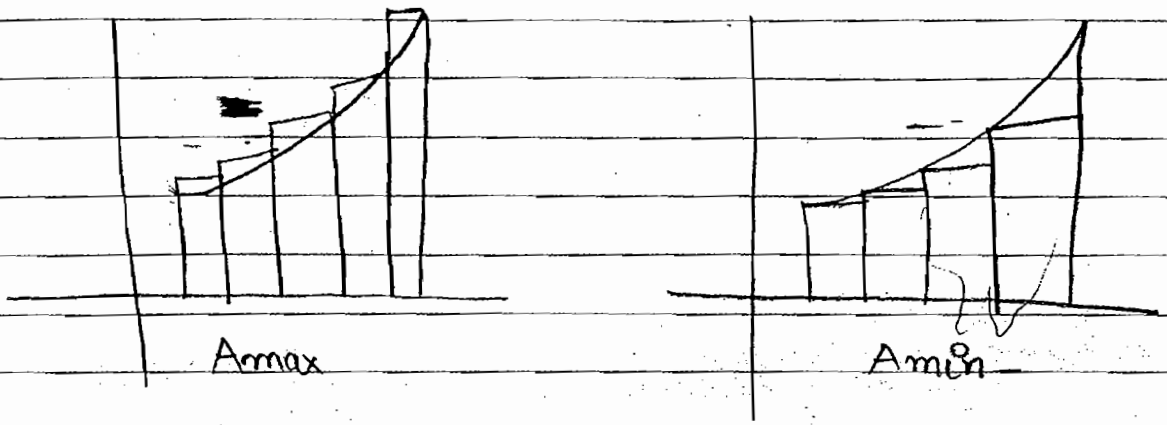
By LMVT $c_n \in (x, x+1/n)$ on f

$\Rightarrow$ $|f'(n) - f'(x)| < \varepsilon$, $n > m$

whenever $|c_n - x| < \delta$; $\frac{1}{n} < \delta$; $n > m$

Exp. $f_n(x) = x^3$, $x \in \mathbb{R}$

RIEMANN - INTEGRATION



Partition of an Interval :→ Let $[a,b] \subset \mathbb{R}$

& $P = \{a = x_0, x_1 \cdots x_n = b\}$

where $x_i \leq x_{i+1}$ $\quad i = 0$ to $n-1$

Then P is called a partition of $[a,b]$ & This P partition $[a,b]$ into n - Subinterval namely $[x_0, x_1]$ $[x_1, x_2] \cdots [x_{n-1}, x_n]$

$$[a,b] = \bigcup_{i=0}^{n-1} [x_i, x_{i+1}]$$

1) $\boxed{\Delta x_r = x_r - x_{r-1}}$ = length of r^{th} sub interval.

2) $\sum_{r=1}^{n} \Delta x_r = b - a$

Norm of P :→ $M(P) = \|P\|$

$$\|P\| = \max \{\Delta x_1, \Delta x_2 \cdots \Delta x_n\}$$

Eg :- $P = \left\{0, \frac{1}{2}, \frac{3}{4}, 1\right\}$

$$\|P\| = \frac{1}{2}$$

Refinement of P :→ Let P & P^* are partition of $[a,b]$

S.t. $P \subseteq P^*$

then P^* is called a refinement of P.

Exp. $P_1 = \{0, \frac{1}{2}, 1\}$

$P_2 = \{0, \frac{1}{2}, \frac{3}{4}, 1\}$

$P_3 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$

$P_1 \subset P_2 \subset P_3$

P_3 is called refinement of P_1 & P_2.

P_2 is called refinement of P_1.

* $P = \{x_0, x_1, \cdots x_n\}$

$P \subsetneq P^*$

$\exists\ t \in P^*$ s.t. $t \notin P$

$\exists\ i = 0$ to $n-1$

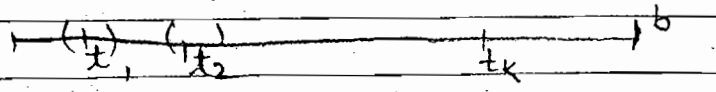
s.t. $t \in (x_i, x_{i+1})$

$[x_i, t]$ $[t, x_{i+1}]$ are two sub interval.

NOTE:- $\boxed{\|P^*\| \leq \|P\|}$

Important Rule :→ Let $t_1, t_2 \cdots t_k \in (a,b)$

Then for a partition of P of $[a,b]$



$t_i^\circ \in (z_i, z_i')$

then $P = \{a, z_1, z_1', z_2, z_2' \cdots z_k, z_k', b\}$

P provides $k+1+k = 2k+1$ sub interval

If P_1 and P_2 are partition of $[a, b]$
then $P_1 \cup P_2 = P$ is also a partition of $[a, b]$

* If P_1 and P_2 are partition of $[a, b]$
for any $\varepsilon > 0$

$$z_i = t_i - \frac{\varepsilon}{2K}$$

$$z_i' = t_i' + \frac{\varepsilon}{2K}$$

$$z_i' - z_i = \frac{\varepsilon}{K}$$

$$\sum \Delta z_i = \varepsilon$$

In any $[a, b]$, finite no. of point if it can be enclosed by into the disjoint intervals of desired desired length in order to given the sum of length of these intervals as less as we pleased".

* $t_1, t_2 \cdots t_{n+1} \cdots \in (a, b)$

$$t_i' \in (z_i, z_i')$$

$$\Delta z_i = \frac{\varepsilon}{2^i}$$

$$\sum \Delta z_i = \sum \frac{\varepsilon}{2^i} = \varepsilon \sum \frac{1}{2^i} = \varepsilon$$

$$\left(\because \frac{1}{2} + \frac{1}{2^2} + \cdots = 1 \right)$$

* Darboux Sum :→

Let f is define and bdd on $[a\ b]$ and

$$P = \{ a = x_0, x_1 \cdots x_n = b \}$$

Then define $m_r = \inf_{x \in [x_{r-1}, x_r]} f(x)$

$M_r = \sup_{x \in [x_{r-1}, x_r]} f(x)$

$$\boxed{m \le m_r \le M_r \le M}$$

Then $\sum m_r \Delta x_r$ & $\sum M_r \Delta x_r$ are defined as Darboux Lower Sum & Darboux upper Sum.

$$L(P, f) = \sum m_r \Delta x_r$$

& $$U(P, f) = \sum M_r \Delta x_r$$

Riemann Sum :→

$$S(P, f) = \sum f(t_r) \Delta x_r \,, \quad t_r \in [x_{r-1}, x_r]$$

Exp. $f : [0, 1] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 1 & , \quad x \in \mathbb{Q} \\ -1 & , \quad x \in \mathbb{Q}^c \end{cases}$$

$$P = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n} \cdots \frac{r}{n}, \frac{n}{n} = 1 \right\}$$

$L(P, f) = ?$ $\quad U(P, f) = ?$

$\Delta x_r = \frac{1}{n} - 0 = \frac{1}{n}$

$\Rightarrow$

$$= \sum \frac{1}{n}$$

$$U(P, f) = 1$$

$$L(P, f) = 1$$

$\sum 1 \cdot \frac{1}{n}$

$= 1 \left(1 + \frac{1}{2} \cdots \right.$

$M_r = \sup_{x \in \left(\frac{r-1}{n}, \frac{r}{n} \right]} \{ x \} = 1$

$$M_r = \sup_{x \in \left[\frac{r-1}{n}, \frac{r}{n}\right]} f(x) = \frac{r}{n}$$

$$U(P,f) = \sum \frac{r}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{r=1}^{n} r$$

$$= \frac{1}{n^2} \cdot \frac{n(n+1)}{2} \Rightarrow \frac{n+1}{2n}$$

$$L(P,f) = \sum \frac{r-1}{n} \cdot \frac{1}{n} \Rightarrow \frac{1}{n^2} \sum_{r=1}^{n} r-1$$

$$= \frac{(n-1)}{2n}$$

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Eg:- $f(x) = x^2 \quad x \in [0,1]$

$$P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots \frac{n}{n} = 1\right\}$$

$$L(P,f) = \sum_{r=1}^{n} \frac{(r-1)^2}{n^2} \cdot \frac{1}{n}$$

$$= \frac{1}{n^3} \sum (r-1)^2 = \frac{1}{n^3} \cdot \frac{n(n-1)(2n-1)}{6}$$

$$= \frac{2}{6}(1-1/n)(1-1/2n)$$

$$= \frac{1}{3}$$

$$U(P,f) = \sum_{r=1}^{n} \frac{r^2}{n^2} \cdot \frac{1}{n}$$

$$\Rightarrow \frac{1}{n^3} \sum_{r=1}^{n} r^2 = \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1}{3}$$

Results & properties :→ If $P \subset P^*$

1) $L(P,f) \leq L(P^*,f)$
2) $U(P,f) \geq U(P^*,f)$
3) $L(P,f) \leq U(P,f) \quad \forall \ P$
4) $\sup_P L(P,f) = \underline{\int_a^b} f(x)\,dx$ (Lower integration)
5) $\inf_P U(P,f) = \overline{\int_a^b} f(x)\,dx$ (Upper integral)

$\|P\| \to 0 \quad L(P,f) = \underline{\int_a^b} f(x)\,dx$

$\|P\| \to 0 \quad , \quad U(P,f) = \overline{\int_a^b} f(x)\,dx$

Condition for Integrability :→

Let f is defiend & bounded on $[a,b]$
f is said to be Riemann Integrable over $[a,b]$ if

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$$

Eg → $f(x) = x \quad , \quad x \in [0,1]$

$U(P,f) = \frac{n+1}{2n} \quad , \quad n \to \infty \quad U(P,f) = \frac{1}{2}$

$L(P,f) = \frac{n-1}{2n} \quad , \quad \underline{\int_a^b} f(x)\,dx = \frac{1}{2}$

$$\int_0^1 x\,dx = \frac{1}{2}$$

$$\therefore \overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx = \frac{1}{2}$$

$\therefore x \in \mathbb{R}[a,b]$

2) $f \in \mathbb{R}[a,b]$

iff for any $\varepsilon > 0$ $\exists\ \delta > 0$

s.t. $U(P,f) - L(P,f) < \varepsilon$

whenever $\|P\| < \delta$

3) $U(P,f) - L(P,f) = \sum M_r \Delta x_r - \sum m_r \Delta x_r$

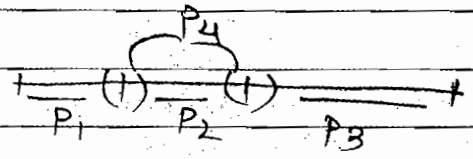
$= \sum (M_r - m_r) \Delta x_r$

This is called oscillatory sum over P.

f is Riemann Integrable if —

$\sum (M_r - m_r) \Delta x_r < \varepsilon$

whenever $\|P\| < \delta$



Exp:-

1. $f(x) = x \quad , x \in [0,1]$

$f \in \mathbb{R}[0,1]$

2. $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{Q}^c \end{cases}$

$f \notin \mathbb{R}[0,1]$

as $\underline{\int_0^1} = -1$ & $\overline{\int_0^1} = 1$

3. $f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \text{ in } (0,1] \\ 0 & \text{else in } [0,1] \end{cases}$

$\sum (M_r - m_r) \Delta x_r = \sum M_r \Delta x_r$

for any $P = \{0 = x_0, x_1 \dots x_n = 1\}$

as $m_r = 0 \quad \forall\ [x_{r-1}, x_r]$

Let $\varepsilon > 0$ be given $\exists\ m \in \mathbb{N}$

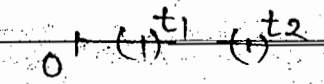
s.t. $\frac{1}{m} < \frac{\varepsilon}{2} \quad \forall\ n \geq m$

$(p \leq q < m)$

In $[0,1]$ the rationals p/q s.t. $q \leq m$ are finite.

(as $p \leq q < m$)

Let they are $t_1, t_2 \ldots t_k$ are those point in $[0,1]$ whose denominator less than q.



Then $\sum (M_r - m_r)\Delta x_r = \sum (M_r - m_r)\Delta y_r + \sum (M_r - m_r)\Delta z_r$

$t_1 \in (z_1, z_1')$ $\quad t_i \in (z_i, z_i')$

$= \sum M_r \Delta y_r + \sum M_r \Delta z_r, \quad \Delta z_r = z_r' - z_r$

$(\because m_r = 0)$

If $x \in [y_{r-1}, y_r]$ $\quad (q \leq m$ out this series$)$

$f(x) = 0$ either $\frac{1}{q} < \varepsilon$ $\quad \left(\begin{matrix} q \geq m \\ \text{rational} \end{matrix}\right)$

$M_r \leq \varepsilon/2$

& $m_r \leq 1 \quad \forall\ p$

$\sum M_r \Delta x_r = \sum M_r \Delta y_r + \sum M_r \Delta z_r < \frac{\varepsilon}{2} \sum M_r \Delta y_r$

$< \frac{\varepsilon}{2} \sum 1 \cdot y_r + \sum 1 \cdot \Delta z_r \quad (\Delta y_r \geq 1)$

$(\because b - a = 1)$

$< \varepsilon$

Let $\Delta z_r = \varepsilon/2K$

$(z_1, z_1') = \left(t_1 - \frac{k}{4k},\ t_1 + \frac{k}{4k}\right) = \frac{\varepsilon}{2k} = \varepsilon$

Riemann function is Riemann integrable.

$M_r = m_r = 1$

$\boxed{\int_0^1 f(x)\,dx = 0}$ $\quad (\because m_r = 0 \ \therefore \int = 0)$

(fun^c on Interval is discontinuous but this fun^c is Riemann-Integrable)

It is Non-neg function.

* Non-neg fun^c may have Integrable 0.

Rules → In any other following condition for a fun^c
f defined & bounded on $[a, b]$
$f \in R[a, b]$

1.) f is Monatonic, Nat conversly. (Monatonic → $R[a,b]$)
Exp. $\sin x \quad x \in [0, \pi]$ (popcorn fun^c)
Nat monatonic.

2) f is continuous on $[a, b]$. but not conversly
(Popcorn fun^c)

3) $[x_{r-1}, x_r]$ for any $\varepsilon > 0$ $\exists\ \delta > 0$
S.t. $|f(t_1) - f(t_2)| < \varepsilon / b-a$
$\forall \quad t_1, t_2 \in [x_{r-1}, x_r]$
S.t. $|t_1 - t_2| < \delta$

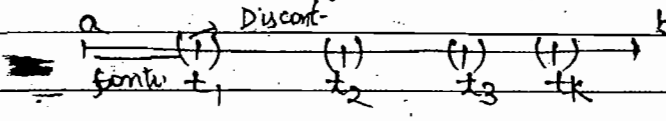
$\sum (M_r - m_r)\, \Delta x_r$
$= \sum |f(t_r) - f(t_r')|\, \Delta x_r$
$= \frac{\varepsilon}{b-a} \cdot b-a = \varepsilon$

$f : [0, 1) \to R$
$f(x) = \begin{cases} 1/x & x \in (0,1) \\ 0 & x = 0, 1 \end{cases}$
Not $R[0,1)$

3) If f is continuous on $[a, b] - S$
→ where $S \subset [a, b]$
then $f \in R[a, b]$
if (a) S is finite.
(b) S is countably infinite.
(c) S' is countable.
→ S is countable

4) $\{x\}$ is $\mathbb{R}[a,b]$

This fun^c has only finite point of discontinuity.



$$\sum (M_r - m_r)\Delta x_r = \sum (M_r - m_r)\Delta y_r + \sum (M_r - m_r)\Delta z_r$$
$$< \frac{\varepsilon}{2} + (M-m)\sum \Delta z_r$$
$$< \frac{\varepsilon}{2} + (M-m)\cdot \frac{\varepsilon}{2(M-m)}$$
$$= \varepsilon$$

Derived Set of popcorn fun^c is uncountable.

(If S' is uncountable then $\mathbb{R}\cdot[a,b]$ only of S is uncountable)

* Algebra of Riemann Integrability :→

If $f, g \in \mathbb{R}[a,b]$ then —

1) $\alpha f + \beta g \in \mathbb{R}[a,b]$
2) $f \cdot g \in \mathbb{R}[a,b]$ ⟶ $\left((f+g)^2 = (f-g)^2 + 4fg\right)$
3) $|f| + |g| \in \mathbb{R}[a,b]$
4) $\max\{f,g\}$, $\min\{f,g\}$
5) $\frac{f}{g}$ provided defined on $[a,b]$ & bounded.

Exp:→ $f(x) = 1 \quad x \in [0,1]$

$$g(x) = \begin{cases} x & x \in [0,1] \\ 1 & x = 0 \end{cases}$$

$$h = \frac{f}{g} = \begin{cases} 1/x & , \ x \in [0,1] \\ 1 & , \ x = 0 \end{cases}$$

h is defined on $[0,1]$ but not bounded.

Exp. $\{x\} = x - [x]$ is $\mathbb{R}[a,b]$

(Mono. − Mono.)

Sum & diff. R-Integrable is again R-integrable.

Exp. $$f(x) = \begin{cases} 1/q & x = p/q \\ 1/q^2 & x = \sqrt{p/q} \\ 0 & \text{else} \end{cases}$$

$f: [0,1] \longrightarrow \mathbb{R}$

This func is $\mathbb{R}[0,1]$

It is continuous.

Exp. $$f(x) = \begin{cases} x, & x \in Q^c\ [0,1] \text{ & } x = 0 \\ p \sin 1/q, & x = p/q \in (0,1] \end{cases}$$

$f \in \mathbb{R}[0,1]$

possible point of discontinuity are countable

So this func is Riemann-Integrable.

Exp :- $f: [0,2] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} x & x \in Q \cap [0,2] \\ x^2 & x \in Q^c \cap [0,2] \end{cases}$$

$f \notin \mathbb{R}[0,2]$

Soln: $P = \left\{0, \frac{2}{n}, \frac{4}{n}, \cdots, \frac{2n}{n}\right\}$

$$\Delta x_r = \frac{2}{n}$$

$$M_r = \frac{2r}{n}, \quad m_r = \frac{2(r-1)}{n}$$

$$\sum M_r \Delta x_r = \sum \left(\frac{2r}{n} \cdot \frac{2}{n} \right)$$

$$= \frac{4}{n^2} (1 + 2 + \cdots n) = \frac{4}{n^2} \cdot \frac{n(n+1)}{2}$$

$$= \frac{2}{n^2} \cdot n^2 \left(1 + \frac{1}{n}\right) = 2$$

$$\sum m_r \cdot \Delta x_r = \sum \frac{2(r-1)}{n} \cdot \frac{2}{n}$$

$$= \frac{4}{n^2} (0 + 1 + \cdots n)$$

$$= \frac{4}{n^2} \cdot \frac{n(n-1)}{2} = 2.$$

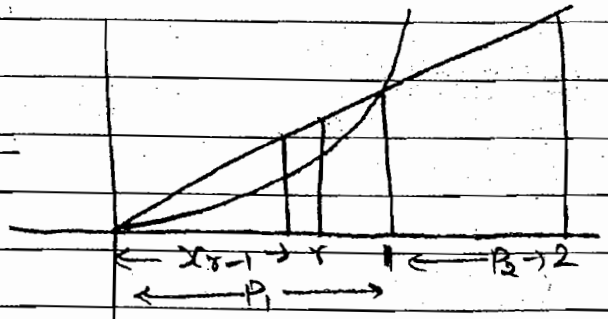
at x^2

$$\sum \left(\frac{4r^2}{n^2} \cdot \frac{2}{n} \right)$$

$$= \frac{8}{n^3} (1^2 + 2^2 + \cdots n^2) = \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{2 \cdot 4}{3} \left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{2n}\right)$$

$$= \frac{8}{3} \text{ //.}$$



$P = P_1 \cup P_2$

$M_r = \text{Sup } f = x_r$
$x \in [x_{r-1}, x_r]$

$m_r = \text{Inf } f = x_{r-1}^2$
$\Rightarrow [x_{r-1}, x_r]$

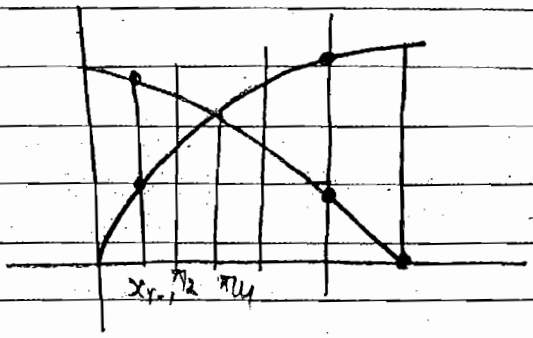
$$\sum M_r \Delta z_r = \sum M_r \Delta x_r + \sum M_r \Delta y_r$$

$$= \sum m_r \Delta x_r + \sum y_r^2 \Delta y_r$$
$$[0\,1] \qquad [1\,2]$$

$$\sum m_r \Delta z = \sum x_{r-1}^2 \Delta x_r + \sum y_{r-1}^2 \Delta y_r$$

$$\underline{\int} = \frac{1}{3} + \frac{3}{2} = \frac{2+9}{6} = \frac{11}{6}$$

Exp. $f(x) = \begin{cases} \sin x & x \in [0, \pi/2] \cap Q \\ \cos x & x \in [0, \pi/2] \cap Q^c \end{cases}$



$M_r = \cos x_{r-1}$

$m_r = \sin x_{r-1}$

$\sum M_r \Delta x_r = \sum_{[0,\pi/4)} \cos x_{r-1} \Delta x_r + \sum_{(\pi/4, \pi/2]} \sin y_r \Delta y_r$

$\overline{\int} = \int_0^{\pi/4} \cos x\,dx + \int_{\pi/4}^{\pi/2} \sin x\,dx$

$\underline{\int} = \int_0^{\pi/4} \sin x\,dx + \int_{\pi/4}^{\pi/2} \cos x\,dx$

This fun^c R ~~$\notin$~~ $\in$ $[0, \pi/2]$

* Application of Integrability :→

1. $f \in \mathcal{R}[a,b]$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^a f(x)\,dx \quad \forall\ c \in (a,b)$$

2. $f(x) \geq 0 \quad \forall\ x \in [a,b]$

& $\int_a^b f(x)\,dx = 0$

$\Rightarrow$ $f(c) = 0$ if f is continuous at $x = c \in (a,b)$

3. f is continuous on $[a,b]$

& $\int_a^b [f(x)]^2\,dx = 0 \Rightarrow f \equiv 0$

$\left(\frac{1}{2}f(c) < f(a) < \frac{3}{2}f(c) \quad \forall x \in (c-\delta, c+\delta)\right.$

is conti. at $x=c$ & $f(c) > 0$

$\exists\ \delta > 0$ s.t. $f(x) > 0 \quad \forall\ x \in (c-\delta, c+\delta)$

$$\int_a^b f(x)\,dx = \int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c+\delta}^{b} \qquad \int_a^b dx = b-a$$

$$= \int_{c-\delta}^{c+\delta} f(x)\,dx$$

$$> \int_{c-\delta}^{c+\delta} \frac{1}{2} f(c)\,dx = \frac{1}{2} f(c) \int_{c-\delta}^{c+\delta} dx$$

$$= \frac{1}{2} f(c) \cdot 2\delta \Rightarrow f(c) \cdot \delta \Rightarrow \int_a^b f(x)\,dx > 0 \quad \text{✗}$$

at rational fun^c must be continuous on popcorn)

If $f \in \mathbb{R}[a,b]$

$f(x) = 0 \quad \forall\ x \in [a,b] - S$

where S is finite or countably Infinite

$$\int_a^b f(x)\,dx = 0$$

(Enclosed point → conti fun^c)

$$\sum m_r \Delta z_r = \sum m_r \Delta x_r + \sum m_r' \Delta y_r$$

$$= 0 + (<\varepsilon) \text{ i.e. } 0$$

$$= 0$$

Let f & g are defined & bdd on $[a,b]$

f is $\mathbb{R}[a,b]$ & $f(x) = g(x) \quad \forall\ x \in [a,b] - S$

where $S \subset [a,b]$

If S is countable then g is Riemann Integrable.

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$$

$$h(x) = f(x) - g(x) \Rightarrow h(x) = \left\{ \begin{matrix} 0, & x \in [a,b] - S \\ \text{unknown} & \end{matrix} \right\}$$

h is conti. on $[a,b] - S$

Exp. $f(x) = \begin{cases} \sin 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$, $x \in [-1, 1]$

only at point 0, $f(x)$ is discontinuous.

$\therefore f(x) \in \mathbb{R}[-1, 1]$

* 1) Let $f \in \mathbb{R}[a, b]$

define $F(x) = \int_a^x f(t)\,dt$ is called indefinite integral of f.

then F is continuous on $[a, b]$

may not be diff. on (a, b)

$$|F(x+h) - F(x)| = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_a^{x+h} + \int_x^a = \left| \int_x^{x+h} f(t)\,dt \right|$$

$$|F(x+h) - F(x)| \leq Mh < \varepsilon, \qquad h = \frac{\varepsilon}{M} < \delta$$

Exp. $f(x) = \begin{cases} -1 & -1 \leq x < 0 \\ 1 & 0 \leq x \leq 1 \end{cases}$ $\left\{ F(x) = \begin{cases} x, \\ -x, \end{cases} \right.$

– (x is not defined on 0)

2) If f is continuous on $[a, b]$

then $F(x) = \int_a^x f(t)\,dt$ is diff. on (a, b)

$\Rightarrow$ & $F'(x) = f(x)$

Proof →

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| < \varepsilon$$

F is called a primitive of f or antiderivative of f.

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f is continuous on $[a, b]$

$c \in [a, b]$

$$F(x) = \int_a^x f(t)\,dt$$

$$F(c+h) - F(c) = \int_a^{c+h} - \int_a^c$$

$$\frac{F(c+h) - F(c)}{h} = \frac{1}{h}\left[\int_a^{c+h} f(t)\,dt\right]$$

$$\Rightarrow \frac{F(c+h) - F(c)}{h} - f(c) = \frac{1}{h}\int_c^{c+h} f(t)\,dt - f(c)$$

$$= \frac{1}{h}\left[\int_c^{c+h} f(t)\,dt - hf(c)\right]$$

$$= \frac{1}{h}\left[\int_c^{c+h} f(t)\,dt - f(t)\int_c^{c+h} dt\right]$$

$$\Rightarrow \left|\frac{F(c+h) - F(c)}{h} - f(c)\right| = \left|\frac{1}{h}\left[\int_c^{c+h}\left(f(t) - f(c)\right)dt\right]\right|$$

$$\leq \frac{1}{h}\int_c^{c+h} |f(t) - f(c)|\,dt$$

$$\leq \frac{1}{h}\cdot\varepsilon\cdot h \Rightarrow \leq \varepsilon$$

Note → $F(x) = \int_a^x f(c)\,dt$ f is conti. on $[a, b]$

$$\Rightarrow \boxed{F'(x) = f(x)}$$

Exp. $f : [-1, 1] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 2x \sin 1/x - \cos 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$\longrightarrow$ cts.

RESULT :-

1) $F(x) = \int_a^x f(t)\,dt$

→ F is cont. on $[a, b]$

→ F may not be diff.

→ F is diff if f is cont.

$f \in IR[a, b]$, f does not hold IVP then f is never differential or primitive.

2) F is diff. on (a, b), conti. on $[a, b]$

$F'(x) = f(x)$, $x \in (a, b)$

Exp. $f = \sqrt{x}$, $x \in [0, 1]$

It is not IR integrable.

If diff. on closed interval then $IR[a, b]$

$f \in IR[a, b]$

then $\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$ → applying LMVT on any $[x_{r-1}, x_r]$

3) Let $f \in IR[a, b]$

$P = \{a = x_0, x_1 \cdots x_n = b\}$

$m \le m_r \le M_r \le M$

$\Rightarrow m\Delta x_r \le m_r \Delta x_r \le M_r \Delta x_r \le M\Delta x_r$

$\Rightarrow \sum m\Delta x_r \le \sum m_r \Delta x_r \le \sum M_r \Delta x_r \le \sum M\Delta x_r$

$\Rightarrow \boxed{m(b-a) \le L(P, f) \le U(P, f) \le M(b-a)}$

as $\|P\| \to 0$ or $n \to \infty$

$m(b-a) \le \int_a^b f(x)\,dx \le \int_a^b f(x)\,dx \le M(b-a)$

$\Rightarrow \boxed{m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)}$

$$\boxed{m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M}$$

If $\mu = \frac{1}{b-a}\int_a^b f(x)\,dx = \mu \Rightarrow \boxed{m \le \mu \le M}$

$\Rightarrow \boxed{\int_a^b f(x)\,dx = \mu(b-a)}$

If f is continuous on $[a,b]$
& as $m \le \mu \le M$
by IVP of continuous function
$\exists\ c \in [a,b]$
S.t. $\boxed{f(c) = \mu}$

* $\frac{b-a}{1+b^2} \le \tan^{-1}b - \tan^{-1}a \le \frac{b-a}{1+a^2}$

$f(x) = \frac{1}{1+x^2} \qquad x \in [a,b]$

$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

* $\frac{b-a}{\sqrt{1-a^2}} \le \sin^{-1}b - \sin^{-1}a \le \frac{b-a}{\sqrt{1-b^2}}$

Q. f & g are conti on $[a\ b]$
$\Rightarrow \int_a^b f(x)\,dx = \int_a^b g(x)\,dx$

Show that $\exists\ c \in (a,b)$
S.t. $f(c) = g(c)$

Sol^n → $h(x) = f(x) - g(x)$
→ Conti.

applying Rolle's Thm

$$H(x) = \int_a^x h(t)\,dt$$

$H(x) \Rightarrow H'(x) = h$ —

$H'(a) = 0, \quad H'(b) = 0$

$$H(a) = \int_a^a h(t)\,dt = 0$$

$$H(b) = \int_b^b h(t)\,dt = 0$$

H is cont. & diff. on $[a, b]$

& $H(a) = H(b)$

$\exists\ c \in [a, b]$

s.t. $H'(c) = h(c) = 0$

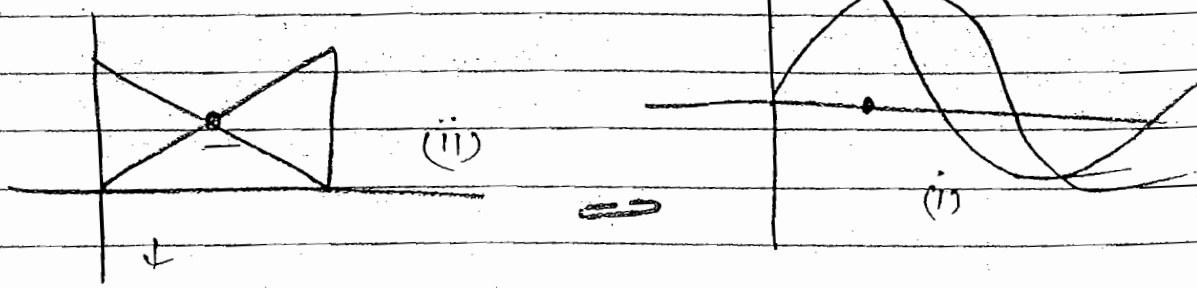
$f(c) - g(c) = 0 \quad \Rightarrow \quad \boxed{f(c) = g(c)}$

Q. (1). $f(x) = \begin{cases} \cos x & x \in [0\ \pi/2] - \{\pi/4\} \\ 0 & x = \pi/4 \end{cases}$

$f \in \mathbb{R}[0, \pi/2]$

$g(x) = \sin x \quad x \in [0, \pi/2]$

f & g are $\mathbb{R}$ on $[0, \pi/2]$



Q. (2) $f(x) = x \quad , x \in [0, 1]$

$$g(x) = \begin{cases} 1-x & , \quad x \neq 1/2 \\ 0 & , \quad x = 1/2 \end{cases} \quad , \quad x \in [0, 1]$$

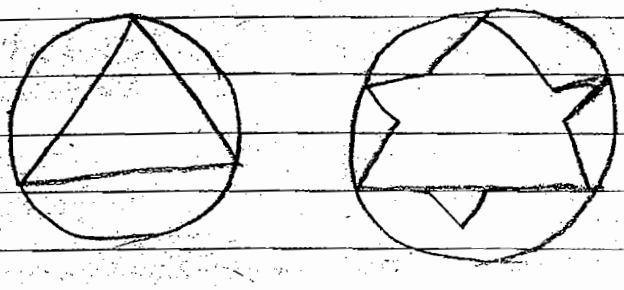
V. Imp

Q. $f(x) = \begin{cases} x \sin \pi/x & x \neq 0 \\ 0 & x = 0 \end{cases}$

$f : [0,1] \longrightarrow \mathbb{R}$

$P = \left\{ 0, \dots \frac{2}{2n-1}, \dots \frac{2}{5}, \frac{2}{3}, 1 \right\}$

Function of Bounded variation :→



Area is not greater than circle.
Length Increase.
Sum of length = $\frac{1}{n}$
→ Infinite

Variation of function :→ Let f is defined on $[a,b]$ and $P = \{a = x_0, x_1, \dots x_n = b\}$ be a partition of $[a,b]$
then $\sum |f(x_r) - f(x_{r-1})|$ is called variation of f over P.
and denoted by $V(f,P)$

1) $|f(b) - f(a)| = |f(b) - f(x_0) + f(x_0) - f(a)|$
for any $x_0 \in (a,b)$
$\leq |f(x_0) - f(a)| + |f(b) - f(x_0)|$
$\leq V(f,P)$
Where $P = \{a, x_0, b\}$

2) $\Rightarrow V(f,P) \leq V(f,P^*)$

3) $\sup_P V(f,P) = \|P\| \to 0 \; V(f,P)$
= The total variation of f on $[a,b]$
$= V(f,[a,b])$

Function of Bounded variation :→ f is said to be B.V. if total variation is finite.

i.e. $v(f,[a,b]) < \infty$

In above Example,

$$v(f,P) = \frac{2}{2n-1}, \frac{2}{2n-3} \frac{-2}{2n-1} = \frac{2(2n-1-2n+3)}{(2n-1)(2n-3)}$$

$$= \frac{4}{(2n-1)(2n-3)}$$

$$\sum |f(x_r) - f(x_{r-1})|$$

$$= \left| f\left(\frac{2}{2n-1}\right) - f\left(\frac{2}{2n+1}\right) \right| + \left| f\left(\frac{2}{2n-3}\right) - f\left(\frac{2}{2n-1}\right) \right|$$

$$+ \cdots + \left| f\left(\frac{2}{3}\right) - f\left(\frac{2}{5}\right) \right| + \left| f(1) - f\left(\frac{2}{3}\right) \right|$$

$$= \left(\frac{2}{2n-1}\right) - 0 + \frac{2}{2n-3} + \frac{2}{2n-1} + \frac{2}{2n-3} + \frac{2}{2n-5} \cdots \frac{2}{3} + \frac{2}{5} + \frac{2}{3}$$

$$\Rightarrow 4\left[\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right]$$

$$= 4 \sum \frac{1}{(2n-1)} \quad ; \quad \frac{4}{2} \sum \frac{1}{(n-1/2)} \longrightarrow \text{Divergent } P=1$$

$$\text{Sup } v(f,P) \geq \lim_{n\to\infty} \sum \frac{1}{(2n-1)}$$

(If it is convergent then B.V.

$\text{Sup}_P v(f,P)$ is ∞

$v(f,[0,1]) \neq \infty \implies \implies$

This fun^c is not bdd variation.

through it is conti., U.C., Riemann.

Condition on B.V. :→

1) If f is monotonic on $[a,b]$ then B.V.

Not conversely.

2) f is Increasing

$P = \{ a = x_0, x_1, x_2 \cdots x_n = b \}$

$v(f, P) = \sum |f(x_r) - f(x_{r-1})|$

$= \sum f(x_r) - f(x_{r-1})$

$= f(b) - f(a)$

$\text{Sup}_P (f, P) = v[f, [a,b]) = f(b) - f(a)$

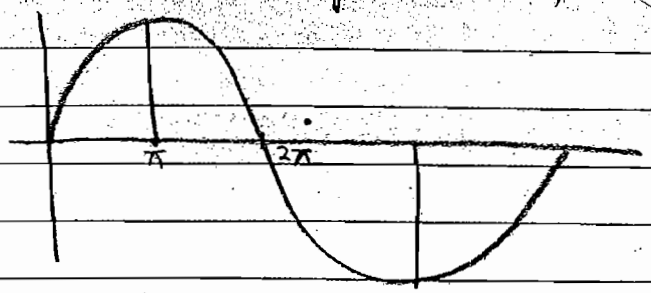
2) If f is of B.V. then it is Bounded, But not Conversly true.

3) for any $x \in (a,b)$

$v(f, [a,c]) + v(f[c,b]) = v(f, [a,b])$

where f is B.V.

Q. Total variation of $\sin x$, $x \in [0, 2\pi]$



$v(\sin x\ [0, 2\pi]) = 4$

Q. $f(x) = 3x^3 - 2x^2 + 5x + 1$, $x \in [0, 5]$

Every polynomial is of B.V. in any $[a,b]$ is Rdd & Continuous. ⟹

$f'(x) = 9x^2 - 4x + 5 = 0$ It has no real root

$x = \frac{+4 \pm \sqrt{16 - 180}}{18}$

$f'(x) > 0$, $f(x) \uparrow$

$v[f[a,b]] = f(5) - f(0) = 125 \times 3 - 50 + 25 + 1 - 1$

$= 375 - 25 = 350$.

4) If f is defined on $[a,b]$
$f'(x)$ exist $\forall\ x \in (a,b)$
$|f'(x)| \leq M$
then f is B.V.

Soln:- By applying LMVT, on $[x_{r-1}, x_r]$

$$\left|\frac{f(x_r)-f(x_{r-1})}{\Delta x_r}\right| = |f'(t_r)| \leq M$$

$$\Rightarrow f(x_r) - f(x_{r-1}) \leq M \Delta x_r$$

Ex. $f(x) = \sqrt{x}$ $x \in [0,1]$
It is monotonic So B.V.
If f is piecewise monotonic $\Rightarrow f$ is B.V.

* If α, β are two Zero's.
$|f(\alpha)-f(a)| + |f(\beta)-f(\alpha)| + |f(b)-f(\beta)|$

* Let f is continuous on $[a,b]$

$$F(x) = \int_a^x f(t)\,dt$$

F is diff on $[a,b]$ conti. on $[a,b]$

$P = \{a = x_0, x_1, \ldots x_n = b\}$
By LMVT on F for $[x_{r-1}, x_r]$
we have,

$$\frac{F(x_r)-F(x_{r-1})}{\Delta x_r} = F'(t_r) = f(t_r)$$

$$|F(x_r)-F(x_{r-1})| = |F'(t_r)|\,\Delta x_r$$
$$= |f(t_r)|\,\Delta x_r$$
$$\sum |F(x_r)-F(x_{r-1})| = \sum |f(t_r)| \cdot \Delta x_r$$

as $\|P\| \to 0$

$$\boxed{v(f, [a,b]) = \int_a^b |f(x)|\,dx}$$

Indefinite integrals of conti. fun^c are of B.V.

Exp.

$$f(x) = \begin{cases} x^{\alpha} \sin \frac{1}{x^{\beta}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$f : [0,1] \longrightarrow \mathbb{R}$, $\alpha, \beta > 0$

$\Rightarrow$ f is B.V. on $[0,1]$ iff $\alpha > \beta$.

Put $\frac{1}{x^{\beta}} = (2n-1)\frac{\pi}{2} \rightsquigarrow \sum \frac{1}{n^{\alpha/\beta}}$

$x = \left(\frac{2}{\pi(2n-1)}\right)^{1/\beta}$ if $\frac{\alpha}{\beta} > 1$ then convergent

if convergent $\longrightarrow$ B.V.

$\lambda \sum \frac{1}{(2n-1)^{\alpha/\beta}}$ iff $\alpha/\beta > 1$ i.e. $\alpha > \beta$.

Exp.

$$f(x) = \begin{cases} x^2 \sin 1/x^3 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

It is conti, diff, U.C., $\mathbb{R}$-Integrable But not B.V. $\therefore$ $\alpha \not> \beta$.

$\Longrightarrow$

Rule :- 1) Linear combination of function of B.V. is again B.V.

2) If f is monotonic, g is monotonic $f-g$ is B.V.

3) If f is B.V. then it is Riemann Integrable Not conversly.

Define $v(x) = v(f, [a\,x])$, $x \in [a, b]$

Called variation function then —

(i) $v(a) = 0$

(ii) $x_1 \leq x_2 \Rightarrow v(x_1) \leq v(x_2)$ i.e. $v(x) \uparrow$

$v[f[a, x_2]) = v([f[a, x_1]) + v(f[x_1, x_2])$

$v(x_2) = v(x_1) + v(f_1[x_1, x_2])$

$v(x_2) - v(x_1) \geq 0$

(iii) f is of B.V. iff v is bounded then $v(b)$ is the total variation.

(iv) v is conti. iff f is conti.

(v) if $g(x) = v(x) - f(x)$

Then $g(x) \uparrow$

if f is of B.V.

$\Rightarrow f = v - g$

where v & g are monotonic (non-decreasing)

v & g are R-Integrable

& $v - g$ is also R-Integrable

$\therefore f$ will be integrable.

(vi) f is of B.V. iff $f = \phi_1 - \phi_2$

—

$\Longrightarrow$

11/11/14

Function of Two or more variable :

1. $\mathbb{R}^n(\mathbb{R})$ is a vector Space.

a. $\|x-y\| = (\Sigma(x_i-y_i)^2)^{1/2}$

$X = (x_1, \cdots x_n)$

$Y = (y_1, \cdots y_n)$

b. $\langle x, y\rangle = \Sigma x_i y_i$

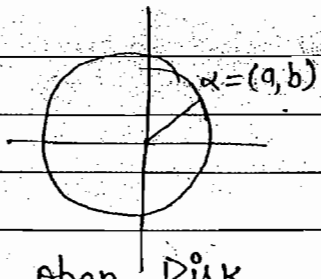
Important Summary :—

1) $\mathbb{R}^n(\mathbb{R})$ is v.s. Then $\mathbb{R}^n(\mathbb{R})$ with (a) is normed linear Space.
with (b) is inner product Space.

nbd :- $D \subseteq \mathbb{R}^2$, $\alpha \in \mathbb{R}^2$

D is a nbd of α

if $\exists\ \delta > 0$ s.t. $\|x-\alpha\| < \delta \subset D$



open Disk

$X = (x, y)$

$\|x-\alpha\| = \sqrt{(x-a)^2+(y-b)^2}$

Limit point :- α is an l.p. of $D \subseteq \mathbb{R}^2$

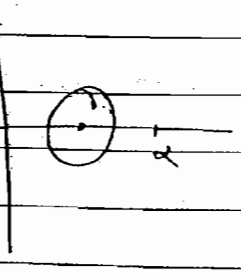
$\forall\ \delta > 0$, $\|x-\alpha\| < \delta \cap D - \{\alpha\} \neq \phi$

$\Rightarrow$

$\Rightarrow$

Every memeber of D is l.p. of D.

$D = \{(x, 0) : -\alpha < x < \alpha\}$



Def :- $f : \mathbb{R}^n \to \mathbb{R}^m$ is defined as function of n - variable where $\mathbb{R}^n$ & $\mathbb{R}^m$ v.s. over R.

1. If $m=1$, f is called real valued fun^c of n-variables.

2. If $f: \mathbb{R}^n \to \mathbb{R}^m$

Then $f(x_1, x_2 \cdots x_n) = (\phi_1, \phi_2 \cdots \phi_m)$

where $\phi_i : \mathbb{R}^n \to \mathbb{R}$

Eg:- $f: \mathbb{R}^2 \to \mathbb{R}^3$

$f(x,y) = (x+y, x-y, x\cdot y)$

$= \phi_1, \phi_2, \phi_3$

$\phi_1 : \mathbb{R}^2 \to \mathbb{R}$

$\phi_1(x,y) = (x+y)$

$\phi_2 : \mathbb{R}^2 \to \mathbb{R}$

$\phi_2(x,y) = x-y$ etc.

$x_1, x_2 \cdots x_n$ are independent

$\phi_1 \cdots \phi_m$ are dependent variables.

Limit :→ $f: D \subset \mathbb{R}^2 \to \mathbb{R}^2$

$\alpha \in D'$, $l \in \mathbb{R}$

we say $\lim_{x \to \alpha} f(x) = l$

if for any $\varepsilon > 0$ $\exists$ $\delta > 0$

s.t. $|f(x) - l| < \varepsilon$

whenever $\|x - \alpha\| < \delta$ in $D - \{\alpha\}$

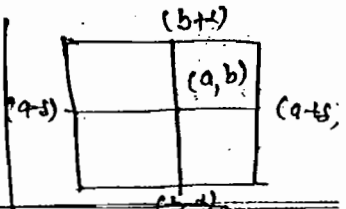
'OR' $\lim_{x \to \alpha} f(x) = l$

iff for any $\varepsilon > 0$ $\exists$ $\delta > 0$

s.t. $|f(x) - l| < \varepsilon$

whenever $0 < |x-a| < \delta$ in $D - \{\alpha\}$

$+ \; 0 < |y-b| < \delta$



Algorithm:- 1) Either l is given or not if not then guess l.

2) Simplify $|f(x)-l|$ using $(2+3+2)^*$ to get $|f(x)-l| \leq c_1|x-a| + c_2|y-b|$ — (i)

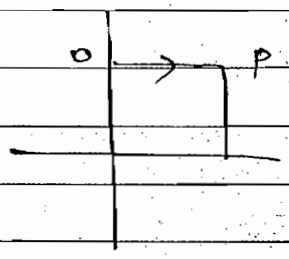
Step-3 or $\leq c_3|x-a|\cdot(y-b)$ — (ii)

Step-3 :- Set $\delta = \frac{\varepsilon}{c_1+c_2}$ for eqn. (i)

Set $\delta = \sqrt{\frac{\varepsilon}{c_3}}$ for (ii)

Examples :-

$\lim_{x\to(1,1)} f(x,y) = ?$



$f(x,y) = 4x+3y$

along OP,

$f(x,y) = 4x+3$ $(\because y=1)$.

along OP, $\lim_{(x,y)\to(1,1)} f(x,y) = \lim_{x\to1}(4x+3) = 7$

$\Rightarrow$ $|4x+3y-7| = |4(x-1)+3(y-1)|$

$\leq 4|(x-1)| + 3|(y-1)|$

$\delta = \frac{\varepsilon'}{7}$

may have to different path, different limit.

NOTE:- For $\lim_{x\to(0,0)} f(x)$

If $\lim_{x\to0} f(x,mx)$ exist and free from m then we say limit exist along ~~radius~~ Radius vector

Then we say "limit exists" a.

its mean (limit may not exist).

Example :— $f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & (x,y) \neq (0,0) \end{cases}$

along radius vector $y = mx$

$$f(x,y) = \frac{mx^3}{x^4+m^2x^2} = \frac{mx}{x^2+m^2}$$

$$\lim_{x\to 0} \frac{mx}{m^2+x^2} = 0$$

find along $y = x^2$

$$f(x,y) = \frac{x^2 \cdot x^2}{x^4+x^4} = \frac{1}{2}$$

$$\lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

Limit Does not exist for $f(x,y)$

Exp: ① $f(x,y) = x \sin\frac{1}{y} + y \sin\frac{1}{x}$, $x \cdot y \neq 0$ ✓

② $f(x,y) = \sqrt{xy}$ ✓

③ $f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$ $(x,y) \neq (0,0)$ ✓

(4) $f(x,y) = \frac{x^2-y^2}{x^2+y}$, $(x,y) \neq (0,0)$ ×

(5) $f(x,y) = \begin{cases} 1 & x \in \mathbb{Q}, y \in \mathbb{Q}^c \\ 2 & x \in \mathbb{Q}, y \in \mathbb{Q} \\ 3 & x \in \mathbb{Q}^c, y \in \mathbb{Q}^c \\ 4 & x \in \mathbb{Q}^c, y \in \mathbb{Q} \end{cases}$ ×

6) $f(x,y) = xy \sin\frac{1}{x}$ $x \neq 0$ ✓

0 $x = 0$

Soln (3) $y = mx$

$$f(x,y) = \frac{mx^2}{\sqrt{x^2+m^2x^2}} = \frac{mx}{\sqrt{1+m^2}}$$

$$\lim_{x\to 0} = \frac{mx}{\sqrt{1+m^2}} = 0$$

$$y = x^2$$

$$f(x,y) = \frac{x^3}{\sqrt{x^2+x^4}} = \frac{x^2}{\sqrt{1+x^2}}$$

$$\lim_{x\to 0} \frac{x^2}{\sqrt{1+x^2}} = 0$$

$\therefore$ Limit exist.

$$y^2 \le x^2+y^2$$

$$|y| \le \sqrt{x^2+y^2}$$

$$\Rightarrow \frac{|y|}{\sqrt{x^2+y^2}} \le 1 \Rightarrow \left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right|$$

$$= \frac{|x|\cdot|y|}{\sqrt{x^2+y^2}} \le |x| = 1\cdot|x-0| + 0\cdot|y-0|$$

$\boxed{\delta = \varepsilon}$

OR

$$\left|\frac{xy}{\sqrt{x^2-y^2}} - 0\right| = \frac{|x|\cdot|y|}{\sqrt{x^2+y^2}} \le |x|$$

(5) $a_n \to a$, $b_m \to b$

$(a_n, b_m) \to (a\ b)$

$(x,y) \to (a,b)$

$(x,y) \to (0,0)$

$P_1\left(\frac{1}{n}, \frac{1}{\sqrt{P_n}}\right) \to (0,0) \to 1$

$P_2\left(\frac{1}{\sqrt{P_n}}, \frac{1}{n}\right) \to (0,0) \to 1$

Not exist.

Continuity :→

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Generalisation :→ $|(x_1+x_2)\sin x_3 + \sin x_1 - 0|$

$\leq |x_1+x_2| + |x_1|$

$\leq 2|x_1| + |x_2|$

Let $f: \mathbb{R}^n \to \mathbb{R}^m$

Then $f(x_1, x_2 \cdots x_n) = (\phi_1, \phi_2 \cdots \phi_m)$

where $\phi_i : \mathbb{R}^n \to \mathbb{R}$

if $\lim_{X \to a} f(x) = L$

where $X = (x_1, x_2 \cdots x_n)$

$a = (a_1 \cdots a_n)$

$L = (l_1, l_2 \cdots l_m)$

$\Leftrightarrow \lim_{X \to a} \phi_i(x) = l_i$

Eg:- $f(x,y) = (x+y,\ x-y,\ x+y+1)$

$f: \mathbb{R}^2 \to \mathbb{R}^3$

$\lim_{X \to 0} f(x) = \lim_{(x,y) \to (0,0)} f(x,y) = (0,0,1)$

Eg:- $f: \mathbb{R}^2 \to \mathbb{R}^2$

$f(x,y) = \left(\frac{xy}{\sqrt{x^2+y^2}} \quad \frac{x^2y}{x^4+y^2} \right)$, $(x,y) \neq (0,0)$

Limit does not exist at origin. (Not conti)

Continuity of generalised :→ f is continuous at a iff each ϕ_i is conti. at $X = a$

Eg:- $f(x,y) = \left(\frac{xy}{\sqrt{x^2+y^2}} \quad \frac{x^4y}{(x^4+y^2)} \right)$, $(x,y) \neq (0,0)$

It is conti at origin provided $(x,y) = (0,0)$

* $f: \mathbb{R}^2 \to \mathbb{R}^2$

$f(a,b) = l > 0$

$|f(x,y) - l| \leq l/2 \quad , \|X - \alpha\| \leq \delta$

$\frac{l}{2} \leq f(x,y) \leq \frac{3}{2}l$

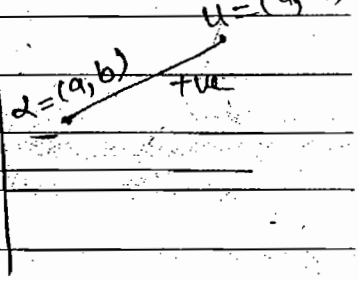
(It retains its Sign)

* **Directional Derivative :→**

$f: D \to \mathbb{R} \quad ; \quad D \subseteq \mathbb{R}^2$

$\alpha, u \in D$

$\alpha = (a,b) \quad , \quad u = (u_1, u_2) \neq (0,0)$



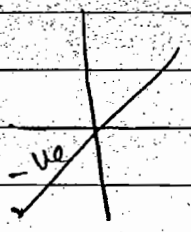
$$\lim_{h \to 0} \frac{f(\alpha + hu) - f(\alpha)}{h} \quad \text{if exist}$$

Then derivative of f in the direction of 'u' at the pt α.

$\alpha + hu = (a,b) + h(u_1, u_2)$

$= (a + hu_1, b + hu_2)$

and denoted by $Df_u(\alpha)$.



NOTE:- When $D \subseteq \mathbb{R}$ then $h > 0$ gives $Df_u(\alpha)$ R.H.D.

$h < 0$ gives $Df_u(\alpha)$ L.H.D.

Exp. $f(x,y) = \begin{cases} \frac{x^2 y}{x^4 + y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$\alpha = (0,0) \quad , \quad u = (u_1, u_2)$

$\alpha + hu = (0 + hu_1, 0 + hu_2)$

$\lim_{h \to 0} (hu_1, hu_2)$ →

$$\lim_{h \to 0} \frac{f(\alpha + hu) - f(\alpha)}{h}$$

$$\lim_{h\to 0} \frac{f(h_1u_1, h_2u_2) - f(0,0)}{h}$$

$$\lim_{h\to 0} = \left(\frac{h_1^2u_1^2 \, h_2u_2}{h_1^4u_1^4 + h_2^2u_2^2}\right) \cdot \frac{1}{h} \quad \text{---}$$

$$= \lim_{h\to 0} \frac{h^2u_1^2u_2}{h^2(h^2u_1^4 + u_2^2)} \Rightarrow \lim_{h\to 0} \frac{u_1^2u_2}{u_1^4h^2 + u_2^2}$$

$$= \frac{u_1^2u_2}{u_2^2} \Rightarrow \frac{u_1^2}{u_2}$$

$$Df_u(a) = \begin{cases} \frac{u_1^2}{u_2} & , u_2 \neq 0 \\ 0 & , u_1 = 0 \end{cases}$$

If $e_i = u = (0, 0 \cdots 1, 0, 0 \cdots)$
1 at i^{th} place.

where $(e_1, e_2 \cdots e_n)$ are standard basis of $\mathbb{R}^n(\mathbb{R})$
Then directional derivative of f at any point a in the direction e_i is called The partial derivative of f along x_i axis.

Then $Df_u(a)$ is denoted by $= f_{x_i}(a) = \frac{\partial f}{\partial x_i}$ at a.

$$f_{x_i}(a) = \begin{cases} 0 & u_2 \neq 0 \\ 0 & u_1 = 0 \end{cases}$$

$\therefore e_1, e_2 = (1,0)$ & $(0,1)$

Exp. $$f(x,y) = \begin{cases} x \sin\frac{1}{y} + y \sin\frac{1}{x} & , x \cdot y \neq 0 \\ 0 & , x \cdot y = 0 \end{cases}$$

at $(0,0)$ f_x & f_y.

$$\lim_{h\to 0} \frac{f(h_1 u_1, h_2 u_2) - f(0,0)}{h}$$

$$= h_1 u_1 \sin \frac{1}{h_2 u_2}$$

$$\lim_{h\to 0} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$\lim_{h\to 0} \frac{h \sin \frac{1}{h} + 0 - 0}{h} = 0$$

$$\lim_{h\to 0} \frac{(0, 0+k) - f(0,0)}{h}$$

Both exist & 0.

$$\lim_{h\to 0} \frac{0 + k \sin 0 - 0}{h} = 0$$

Q. $$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

* **Differentiability :→** $$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \alpha = f'(a)$$

$$\Rightarrow \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \alpha + \phi(h)$$

$$\Rightarrow \lim_{h\to 0} f(a+h) - f(a) = h\phi(h) + \alpha h$$

$$\lim \frac{f(a+h) - f(a)}{h} = \alpha$$

$$\Leftrightarrow \lim_{h\to 0} \phi'(h) = 0$$

$f : \mathbb{R}^2 \to \mathbb{R}^2$

f is said to diff. at (a,b) if $\exists$ constants α and β depending on (a,b) and the fun^{s} f both s.t.

$f(a+h, b+h) - f(a,b) - h\alpha - k\beta$
can be expressed as $\sqrt{h^2+k^2} \cdot \phi_1(h,k)$
(OR)
$h\phi_1(h,k) + k\phi_2(h,k)$
where $\lim_{(h,k)\to(0,0)} \phi(h,k) = 0$

Exp. Rule :- The constant α and β mention in the defination are partial derivative of f, the pt (a,b) in the direction of x & y respectively.

along X-axis , $k = 0$

$$\lim_{h\to 0} \frac{f(a+h,b) - f(a,b) - \alpha h}{h} = \lim_{h\to 0} \frac{|h| \cdot \phi_1(h,0)}{h}$$

$$= 0$$

$$\alpha = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} = \frac{\partial f}{\partial x} \text{ at } (a,b)$$

Exp. $$f(x,y) = \begin{cases} \frac{x^2 y}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ at 0, $\alpha = 0$, $\beta = 0$

By previous Example.

$$f(h+0, k+0) - 0 - 0 = \frac{h^2 k}{\sqrt{h^2+k^2}}$$

$$= \frac{h^2 k}{(h^2+k^2)} \sqrt{h^2+k^2} = \sqrt{h^2+k^2}\,\phi(h,k)$$

This fun^c is diff. at $(0,0)$

$$f(x,y) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x}, & x\cdot y \neq 0 \\ 0, & x\cdot y = 0 \end{cases}$$

$\alpha = 0, \beta = 0$

$f(h+0, k+0) = h\sin\frac{1}{k} + k\sin\frac{1}{x}$

It is not diff.

$f : \mathbb{R}^n \to \mathbb{R}^m, \quad n \geq 1$

f is called fun^c of several variables or n-variable
$m = 1$ real valued else vector valued.

$f(x_1 \cdots x_n) = (\phi_1, \cdots \phi_m)$
where each ϕ_i is real valued fun^c of n-variable.

$\phi : \mathbb{R}^n \to \mathbb{R}$

$u = f(x,y), \quad f : \mathbb{R}^2 \to \mathbb{R}$

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$n = 1$

$f(x) = 0$

$f(x)$

$n > 1$

$\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0$

$du = 0$

Sign of $d^2u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u$

$d^2u = \frac{\partial^2 u}{\partial x^2}(dx)^2 + \frac{\partial^2 u}{\partial y^2}(dy)^2 + 2\frac{\partial^2 u}{\partial x \partial y}dx\,dy$

$= AX^2 + BY^2 + 2HXY$

$AX^2 + BY^2 + 2HXY > 0 \to$ Minima

$AX^2 + BY^2 + 2HXY < 0 \to$ Maxima

Exp. $f(x,y) = x^3 + y^3 - 3axy$, $a>0$

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay \quad \text{——(I)}$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3ax \quad \text{——(II)}$$

$(0,0)$, (a,a) are stationary point

$$\frac{\partial^2 f}{\partial x^2} = 6x = 6a$$

$$\partial^2 f / \partial y^2 = 6y = 6a$$

$$\frac{\partial^2 f}{\partial x \partial y} = -3a$$

$$d^2u = 6ax^2 - 6axy + 6ay^2$$

$$= 6a(x^2 - xy + y^2)$$

$$= 6a\left[x^2 - \frac{1}{2}\cdot 2xy + \frac{y^2}{4} - \frac{y^2}{4} + y^2\right]$$

$$d^2u = 6a\left[\left(x - \frac{y}{2}\right)^2 + \frac{3y^2}{4}\right] > 0$$

at point (a,a), f is minima $(\because a>0)$

$$\Rightarrow a\left[\left(x + \frac{hy}{a}\right)^2 + \frac{by^2}{a} - \frac{h^2y^2}{a^2}\right]$$

$$\Rightarrow a\left[\left(x + \frac{hy}{a}\right)^2 + \frac{y^2}{a^2}(ab - h^2)\right]$$

$$\begin{bmatrix} a & h \\ h & b \end{bmatrix} \qquad a>0 \;,\; ab - h^2 > 0 \implies +ve$$

$$a<0 \;,\; ab - h^2 > 0 \quad -ve$$

12/11/14

Matrix of Derivative :→ Let $f: D \subseteq \mathbb{R}^n \to \mathbb{R}^m$

$c \in D$ & $\exists\ \delta > 0$ s.t. $\|x - c\| < \delta < D$

$f(x_1, \cdots x_n) = (\phi_1, \cdots \phi_m)$

$a_{ij} = \frac{\partial \phi_i}{\partial x_j}$ at c

$Df(c) = [a_{ij}]_{m \times n}$

$$= \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial \phi_m}{\partial x_1} & \cdots & \cdots & \frac{\partial \phi_m}{\partial x_n} \end{bmatrix}_{m \times n}$$

Exp. $f: \mathbb{R}^2 \to \mathbb{R}^3$

$f(x, y) = (x+y, x-y, x \cdot y)$ at $c = (1, 1)$

$$D[f(c)] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$ (Not square matrix)

Exp. $f: \mathbb{R}^2 \to \mathbb{R}^2$

$f(x, y) = (x+y, x \cdot y)$

$$= \begin{bmatrix} 1 & 1 \\ y & x \end{bmatrix} \Rightarrow J_c = (x - y)$$

Exp. $f(x, y) = (\sin x \sin y \cos x \cos y, \sin x \sin y)$

Let $\mathbb{R}^n \to \mathbb{R}^n$

⟹ then at any pt c, Jacobian defined —

$J_c = |Df(c)|$

Thm:- At any pt c, J_c is non-zero then $\exists$ a nbd of c in which f is invertible.

i.e. in nbd of c f is one-one.

Note:- Q:- $f: \mathbb{R}^2 \to \mathbb{R}^2$

$z = x + iy$, $a = a_1 + ia_2$

$f(x, y) = z^2 = (x^2 - y^2, 2xy)$

$$Df(a) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix} = 2\begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix}$$

If $h = h_1 + ih_2$

$$Df(a)h = 2\begin{bmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$= 2\begin{bmatrix} a_1h_1 - a_2h_2 \\ a_2h_1 + a_1h_2 \end{bmatrix} = 2\left[(a_1 + ia_2)\cdot(h_1 + ih_2)\right]$$

$Df(a)h = 2a \cdot h$

f is 1-1 in $\mathbb{R}^2$ ✗

f is 1-1 in $\mathbb{R}^2 - \{0,0\}$ ✓

$J_c = |Df(a)| = 4|a|^2$

It is invertible | 1-1 in $\mathbb{R}^2 - \{0,0\}$

Q:- $f: \mathbb{R}^n \to \mathbb{R}$

$a = (a_1, a_2 \cdots a_n)$ be a fixed vector in $\mathbb{R}$

$f(x_1, x_2 \cdots x_n) = a_1x_1 + \cdots + a_nx_n$

(✗)

(i) $Df(0) = 0$ ✗ $\qquad Df(x) = [a_1 \cdots a_n]_{1\times n}$

$Df(0) = [a_1 \cdots a_n]$

$A = [Df(0)]^T [Df(0)]$

$f: \mathbb{R}^n \to \mathbb{R}$

$f(x_1, x_2 \cdots x_3) = x_1 + 2x_2 + \cdots + nx_n$

$$= \begin{bmatrix} a_1 \\ \vdots \\ \end{bmatrix}[a_1 \cdots a_n]$$

$$\begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix} \begin{bmatrix} 1 & \cdots & n \end{bmatrix}$$

$$\underbrace{n,\ 0 \cdots 0,\ \Sigma n^2}_{n-1 \text{ times}}$$

$$\begin{bmatrix} X = (x_1 \cdots x_n) \\ A = X^T X \rightarrow \text{Symmetric matrix} \\ 0 \cdots 0, \Sigma x_i^2 \\ \cdots \end{bmatrix}$$

(ii) $Df(a)b = a_1 b_1 + \cdots + a_n b_n$

(iii) $Df(a)a = \|a\|^2$

Convex :-

$f : S \longrightarrow \mathbb{R}$, $S \subseteq \mathbb{R}$

f is called convex on S if $\forall x_1, x_2 \in S$

$$f(\lambda x_1 + \mu x_2) \leq \lambda(f(x_1)) + \mu f(x_2)$$

$$\lambda + \mu = 1 \qquad \lambda, \mu \geq 0$$

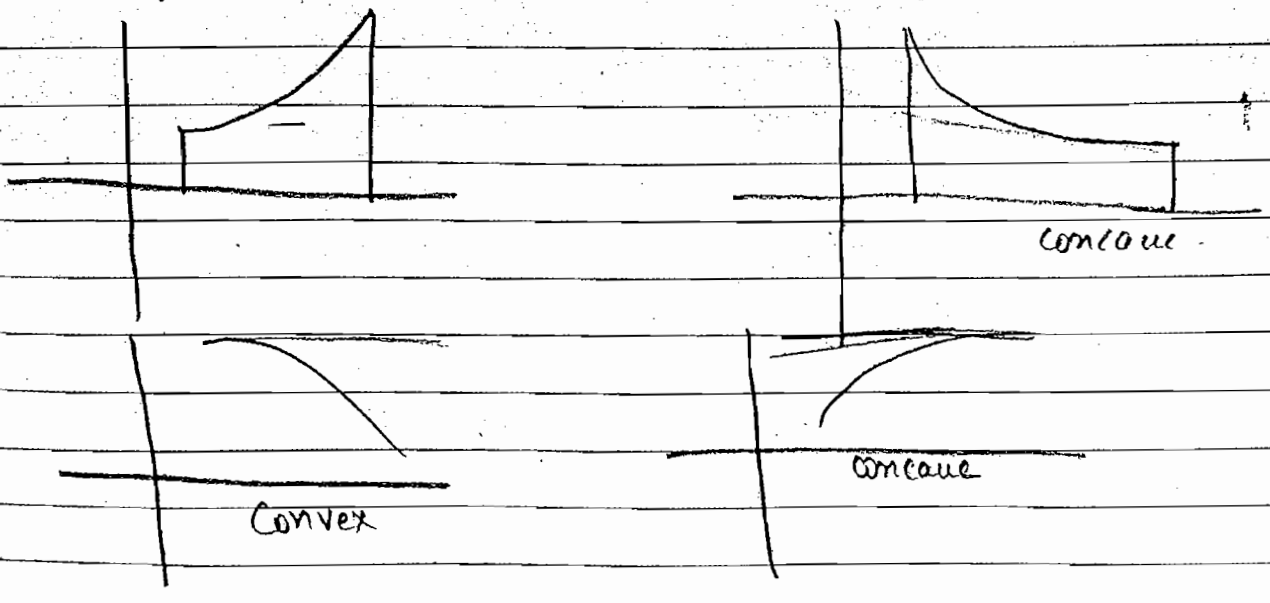
$$f(x_1) = y_1 \qquad f(x_2) = y_2$$

$$f(\lambda x_1 + \mu x_2) \leq \lambda(y_1) + \mu y_2$$

Image of convex combination $\leq$ convex combi. of image.

Result :- If $f \in C_1[a\ b]$

f is convex at $[a, b]$ if at any point c the tangent to the curve lie below the curve.



(3) $f \in C_2[a,b]$

f is convex if $f''(x) \geq 0$

f is concave if $f''(x) \leq 0$

" Constant fun^c are convex and concave "—

$f(x) = |x|$

$|\lambda x_1 + \mu x_2| \leq \lambda|x_1| + \mu|x_2|$

(4) Let $f''(x) = 0$ at $x = \alpha$

& $f''(x) < 0 \quad \forall\ x \in (\alpha-\delta, \alpha)$

$f''(x) > 0 \quad \forall\ x \in (\alpha, \alpha+\delta)$

for some δ

'OR' $f'(x) \Rightarrow f'(x) > 0 \quad , x \in (\alpha-\delta, \alpha)$

$f''(x) < 0 \quad , x \in (\alpha, \alpha+\delta)$

Point of inflection :— A point on at which fun^c & changes its concave & convexity is defined as point of inflexion.

If $f \in C_2[a,b]$

then zeros of f, $f''(x) = 0$ are possible point of inflexion need not to be.

Confirmality comes when $f''(x)$ changes its sign as x passes 'a' (Zeros of $f''(x)$)

Exp $f(x) = x^5$

$f''(x) = 20x^3$ at $\alpha = 0$

$f''(x) = 0$

$\Rightarrow$ '0' is point of inflexion $\begin{pmatrix} L.H.S = -0 \\ R.H.S = 0 \end{pmatrix}$

$\Rightarrow$ $f(x) = x^4$

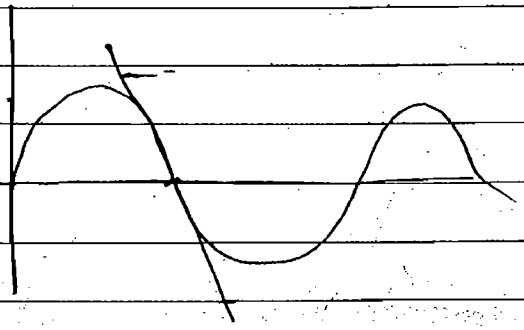
0 is not pt of inflexion

Note:- At pt of Inflexion, the tangent crosses the curve.
i.e. The curve lie on both side of tangent.

Exp. $f(x) = \sin x$

$f''(x) = -\sin x$

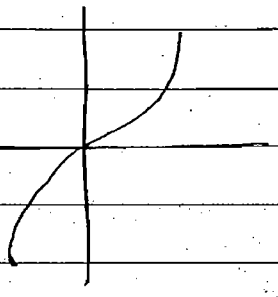
$x = n\pi$



Exp. $f(x) = x^3$

$f''(x) = 6x$

fun^c is change its convex & concavity



$g(x) = \frac{f(x)}{a} \uparrow (0, \infty)$

$f'(x) > 0$ then $f(x) > 0$, we can't say f is Increasing.

$f \in C_2[0,1]$ & f is convex (Non-linear)

$\phi(x) = f(x) + f(1-x)$

Is not monotonic.

$\phi'(x) = f'(x) + f'(1-x)(-1)$

$\phi'(x) = f'(x) - f'(1-x) = 0$

$f'(x) = f'(1-x) > 0$

$\phi''(x) = f''(x) + f''(1-x)$.

$f''(x) = -f''(1-x) < 0$

at $x = 1/2$

$\phi'(x) = 0$

$x \in (0, 1/2)$, $x < 1-x$

$f'(x) < f'(1-x)$

$x \in (1/2, 1)$, $x > 1-x$

at $1/2$ fun^c has local ~~extrema~~ minima.

$\phi'(x) \leq 0 \quad , \quad x \in (0, 1/2)$

$\phi'(x) \geq 0 \quad , \quad x \in (1/2, 1)$

COMPLEX ANALYSIS

Phase-1 (30%)

1) Complex No.
2) Function of complex variable.
3) LCD | Analytic fun^c | Singularities

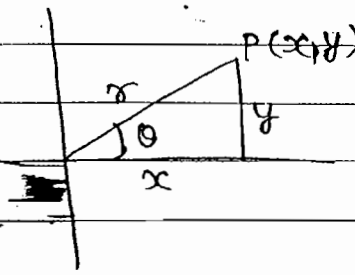
Phase-2 (50%)

4) Complex Integration & Basic thm.
5) Application of complex Integration.
6) Application of Expansion.
7) Residues & Calculus of Residue.
8) Conformal mapping.

Phase-3 (20%)

8) Conformal mapping.
9) Power Series.

UNIT-1



$z = x + iy$

$= re^{i\theta}$ → polar.

where $r = \sqrt{x^2+y^2} = |z|$

$\theta = \text{Arg } z$

Where θ is given by

$$0 < \alpha = \text{Tan}^{-1}\left|\frac{y}{x}\right| < \frac{\pi}{2}$$

$$\theta = \begin{cases} \alpha & x>0, y>0 \\ \pi/2 & x=0, y>0 \\ \pi-\alpha & x<0, y>0 \\ -\pi+\alpha & x<0, y<0 \\ \pi & x<0, y=0 \\ -\pi/2 & x=0, y<0 \\ -\alpha & x>0, y<0 \\ 0 & x>0, y=0 \end{cases}$$

$$-\pi < \theta < \pi$$

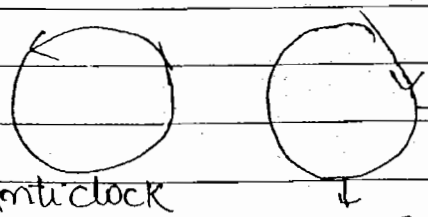
This is called principal argument.

Exp. $z = 1 - 4i$ $\quad x=1, y=-4$

$$\alpha = \text{Tan}^{-1}\left|\frac{-4}{1}\right| = \text{Tan}^{-1}(4) = \frac{76^\circ\pi}{180}$$

$\theta = -\alpha$, $\quad \theta = -\frac{76\pi}{180}$ $\quad (\because x = 1 > 0,\ y = -4 < 0)$

⇒ (clockwise = -ve)



→ clockwise

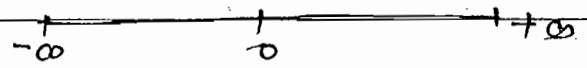
Anticlock

Inside points outside points of circle.

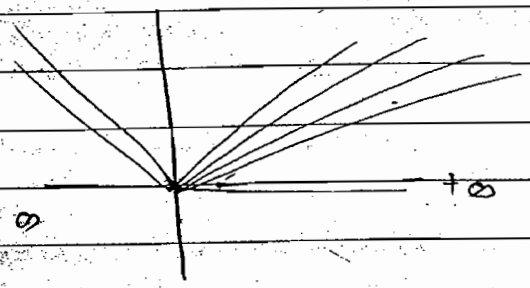
$$\arg z = \{ 2n\pi + \text{Arg } z \; ; \; n \in \mathbb{Z} \}$$

Infinity :→

Stereographic projection :→



(one position one point)



(Infinite position for one point $(\infty, -\infty)$)

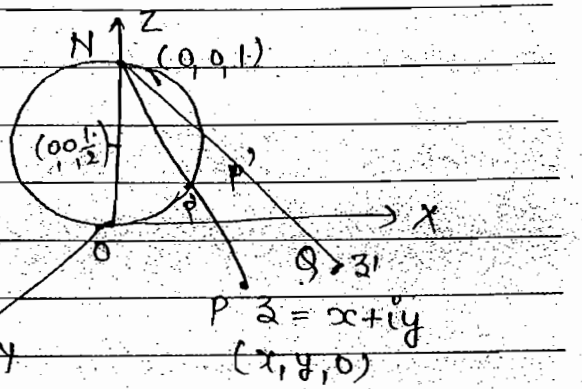
$$S_1 = \left\{ (x,y,z) ; x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4} \right\}$$

↓

Infinite Set

$S_2 = S_1 - \{(0,0,1)\}$

$S_2 = \{ (x,y,0) ; x+iy \in \mathbb{C} \}$



$S_1 \cup S_2$ (Both infinite set) (Infi − fini = Infi) → S_2

$S_2 \cup S_2$ by joining P, Q - 1. etc to N.

Equ^n of line NP

$$\frac{X-0}{x-0} = \frac{Y-0}{y-0} = \frac{Z-1}{0-1} = r \qquad \left(\frac{X-x_1}{x_2-x_1} = \frac{Y-y_1}{y_2-y_1} = \frac{Z-z_1}{z_2-z_1} \right)$$

$$X^2 + Y^2 + (Z - 1/2)^2 = 1/4$$

$X = xr$, $Y = yr$, $Z = -r + 1$

$$x^2r^2 + y^2r^2 + (1 - r - 1/2)^2 = 1/4$$

$$= x^2r^2 + y^2r^2 + r^2 + \frac{1}{4} - r = \frac{1}{4}$$

$$= r^2(x^2 + y^2 + 1) = r$$

$r =$

$$X = \frac{x}{1+x^2+y^2}, \quad Y = \frac{y}{1+x^2+y^2}, \quad Z = \frac{x^2+y^2}{1+x^2+y^2}$$

$$X = x(1-Z), \quad Y = y(1-Z)$$

$$x = \frac{X}{1-Z}, \quad y = \frac{Y}{1-Z}$$

These Equ^n are called Steographic Equ^n.

Exp. $z = 1+i$

$x = 1, \quad y = 1$

$$X = \frac{1}{1+1+1} = \frac{1}{3}, \quad Y = \frac{1}{3}, \quad Z = \frac{2}{3}$$

pt of Sphere $\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$ Ans.

* $z_1, z_2 \in C$

$\chi(z_1, z_2)$ = distance b/w the correspondence of z_1 & z_2 on the sphere.

is called "Chordal" distance.

This Chordal distance $\boxed{\chi(z_1, z_2) \leq 1}$

if $\chi(z_1, z_2) = 1$ then corresponding points are diametically opposite in sign then z_1, z_2 are called "Antipodal points."

Exp. $\chi(1+i, \infty)$,

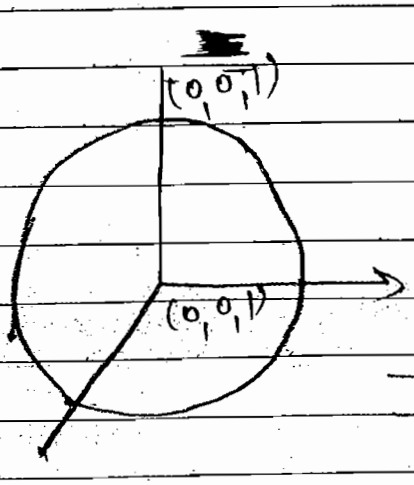
$z_1 = 1+i \longrightarrow (1/3, 1/3, 2/3)$

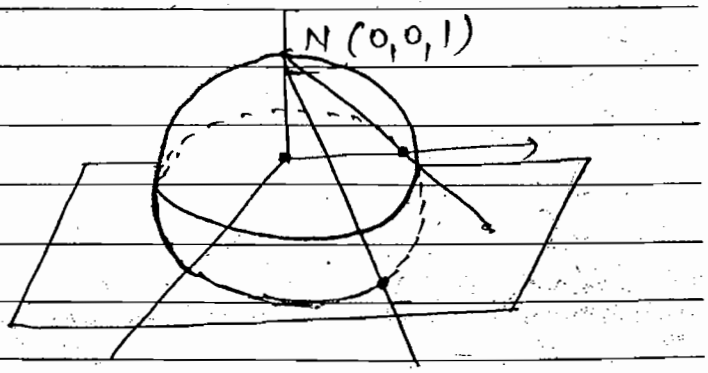
$z_2 = \infty \longrightarrow (0, 0, 1)$

$$= \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}-1\right)^2} = \sqrt{\frac{3}{9}} = \frac{1}{\sqrt{3}}$$

13 | NOV | 14

$$\chi(z,\infty) = \frac{1}{\sqrt{1+|z|^2}} = \frac{1}{\sqrt{1+x^2+y^2}}$$





Inside point cut sphere below axis
Outside " " " above axis

$$x^2+y^2+z^2 = 1$$

(x,y,z) $(x,y,0)$

Eqn $\quad \frac{x-0}{x-0} = \frac{y-0}{y-0} = \frac{z-1}{0-1} = r$

$$X = xr, \quad Y = yr, \quad Z = -r+1$$

$$r^2(x^2+y^2) + (1-r)^2 = 1$$

$$r^2(x^2+y^2) + 1 + r^2 - 2r = 1$$

$$-r(x^2+y^2+1) = 2$$

$$r = \frac{2}{1+x^2+y^2}$$

$$X = \frac{2x}{1+x^2+y^2}, \quad Y = \frac{2y}{1+x^2+y^2}, \quad Z = \frac{x^2+y^2-1}{x^2+y^2+1}$$

$$\Rightarrow$$

$$X = \frac{z+\bar{z}}{1+|z|^2}, \quad Y = \frac{-i(z-\bar{z})}{1+|z|^2}, \quad Z = \frac{|z|^2-1}{|z|^2+1}$$

$$x+iy = \frac{X+iY}{1-Z}$$

Chordal distance $\chi(z_1, z_2) = \dfrac{2|z_1 - z_2|}{\sqrt{(1+|z_1|^2)(1+|z_2|^2)}}$

$$\chi(z, \infty) = \frac{2}{\sqrt{1+|z|^2}}$$

$z_1 \longrightarrow (x_1, y_1, z_1)$

$z_2 = (x_2, y_2, z_2)$

$$= \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$$

$$\chi(z_1, z_2) = \sqrt{2 - 2(x_1x_2 + y_1y_2 + z_1z_2}$$

\# $z + \infty = \infty \quad \forall\ z \in \mathbb{C}$

$\frac{1}{0} = \infty$ $\qquad 0^n = 0 \quad \forall\ n \in \mathbb{N}$

$\bar{0}^n = \infty \quad \forall$ "

$\frac{1}{\infty} = 0$ $\qquad \infty^n = \infty \quad \forall$ "

$0 \cdot \infty = 0$ $\qquad \bar{\infty}^n = 0 \quad \forall$ "

$\infty - \infty$, $\frac{\infty}{\infty}$, $\frac{0}{0}$ are undefined.

" ∞ is called north pole of origin. "

" Every bour unbdd set has atleast one limit pt is "∞" .

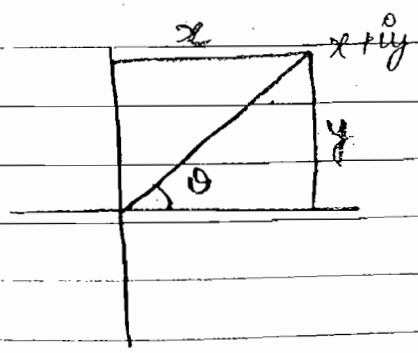
$\Rightarrow |z| > r$ is nbd of ∞.

Argument of z :→

$z = x + iy$

$x = r\cos\theta$

$y = r\sin\theta$



$= \frac{x}{r}$, $\sin\theta = \frac{y}{r}$

of all angles which satisfy this eqn is called arg z }

$z = \theta$

$\text{Arg } z = \{ 2n\pi + \theta : n \in \mathbb{Z} \}$

$-\pi < \text{Arg } z \leq \pi$

ally Equal No. (PEN):

$z_1, z_2 \in \mathbb{C}$ are PEN (pair of)

$|z_1| = |z_2|$

$\arg z_1 = \arg z_2 = 2\pi n$ for some $n \in \mathbb{Z}$

$z = \alpha + i\beta$

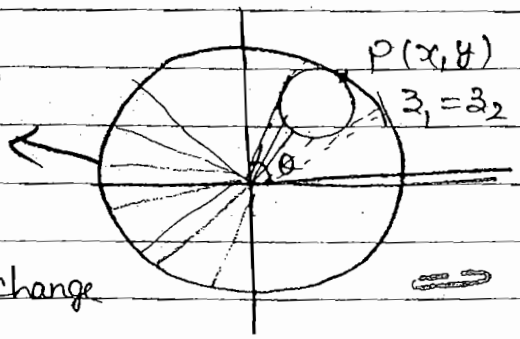
$\arg z =$

$z_1 = re^{i\theta_1}$

$z_2 = re^{i\theta_2}$, $\theta_1, \theta_2 \in \arg z$

and z_2 are PEN (∵ same)

d circle θ change one revolution into $2\pi + \theta$



d circle θ ~~not~~ change

z_2

Let c be a circle not passing through origin

$C: |z - z_0| = r$

le pt z when moves on the curve c after

One complete revolution it occupied the same position and we say at that position pair of we have pair of PEN namely z_1, z_2 and

$\arg z_1 - \arg z_2 = 2\pi$ → Anti clock. Origin is enclosed

$= -2\pi$ → clock

$= 0$ Origin is outside

we have calculating the change in the argument of z after one complete revolution.

FUNCTION OF COMPLEX VARIABLE

⇒ A function from $f: D \subseteq \mathbb{C} \to \mathbb{C}$ is called function of complex variable.

When $z = x + iy$

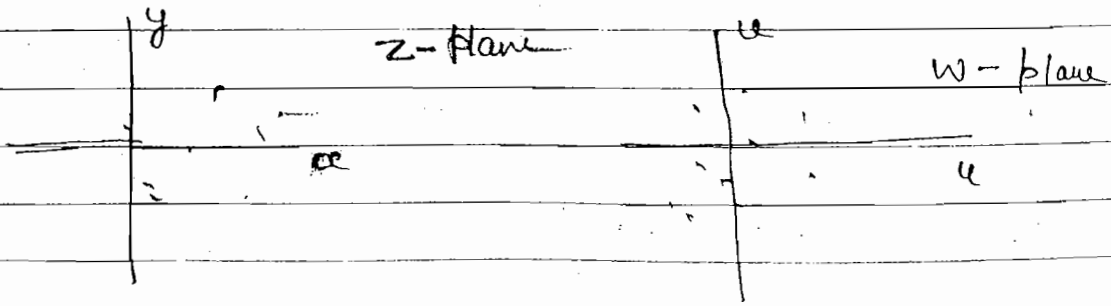
$$w = f(z) = u(x,y) + iv(x,y)$$

where $u: S \subseteq \mathbb{R}^2 \to \mathbb{R}$

$v: S \subseteq \mathbb{R}^2 \to \mathbb{R}$

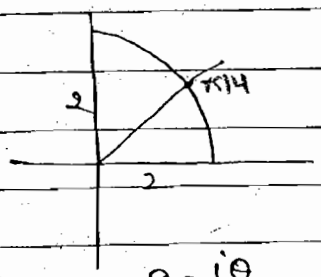
$$S = \{ (x,y) : x+iy \in D \}$$

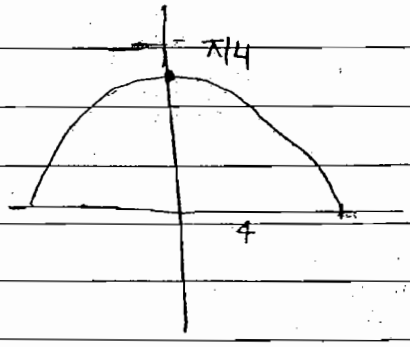
To analysis this fun^c geometrically we take two copies of complex plane & allocate the element of D and their images in this plane respectively we refer these two planes as z plane & w plane.



And this correspondence of the position defines mapping of the func^c $w = f(z)$

Example :- $f(z) = z^2$ $\qquad$ $f(z) = 4e^{2i\theta}$

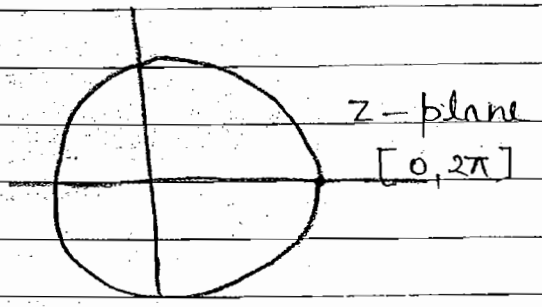


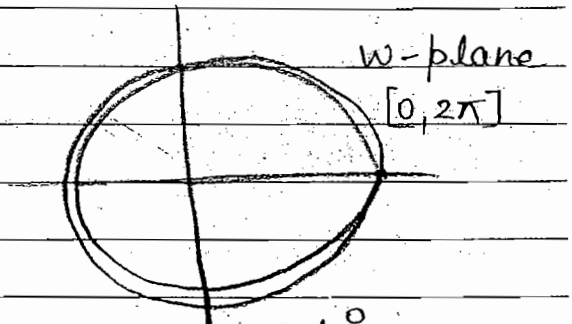


$z = 2e^{i\theta}, \quad \theta \in [0, \pi/2]$

$= 2\cos\theta + 2i\sin\theta$

$z = 2e^{i\theta}, \quad \theta \in [0, 2\pi]$





1 revolution
Change in arg = 2π

2 revolution
Change in arg = 4π

$\arg w = 2 \arg z$

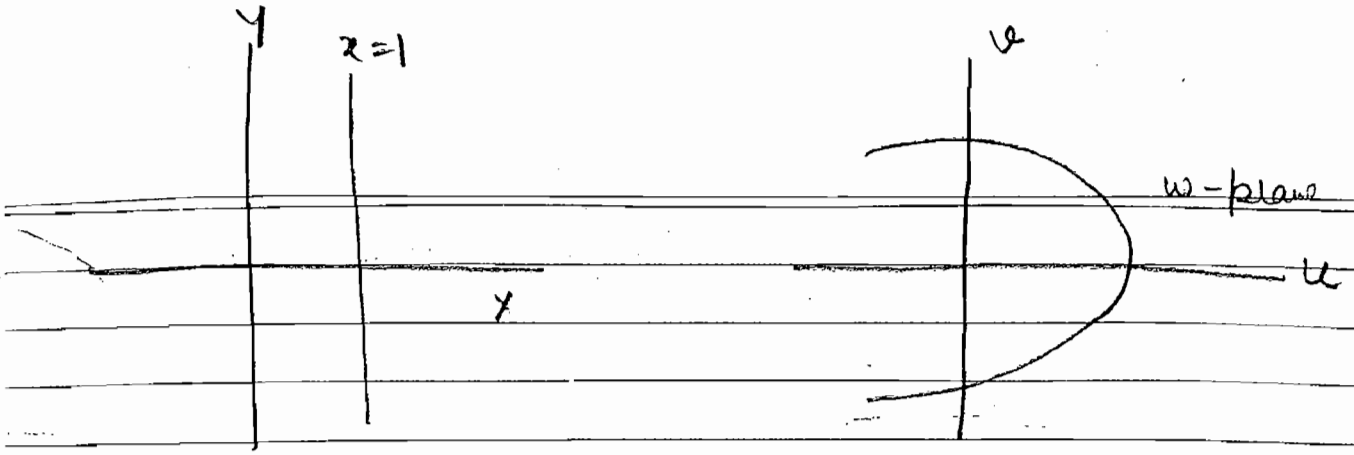
Note :-

When z complete one-revolution in z-plane the change in arg of w where $w = f(z)$ is a function must be multiple of 2π. $\Rightarrow$

* $w = f(z) = z^2 = x^2 - y^2 + 2xyi$

$u = x^2 - y^2$

$v = 2xy$



$x = 1$

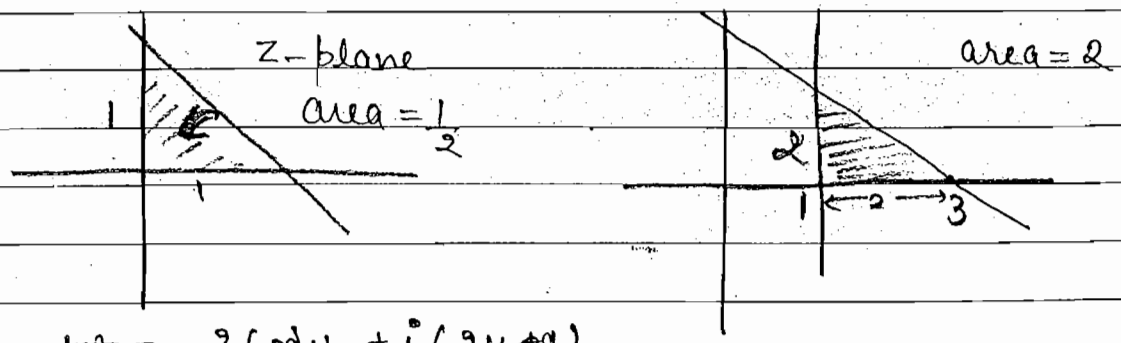
$\therefore u = 1 - y^2, \quad v = 2y \Rightarrow y = v/2$

$u = 1 - \left(\frac{v}{2}\right)^2$

$\frac{v^2}{4} = -(u-1) \Rightarrow v^2 = -4(u-1)$

Exb. $w = 2z + 1$

$x + y \leq 1, \quad x \geq 0, \quad y \geq 0$



$w = 2(x+1) + i(2y+0)$

$u = 2x + 1, \quad v = 2y$

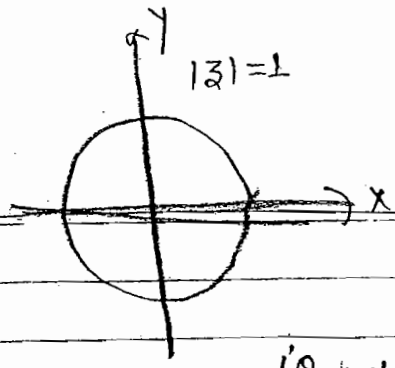
$x = \frac{u-1}{2}, \quad y = \frac{v}{2}$

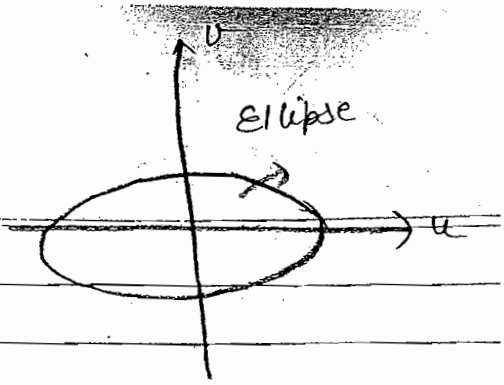
$\Rightarrow \frac{u-1}{2} + \frac{v}{2} \leq 1 \quad \& \quad \frac{u-1}{2} \geq 0, \quad \frac{v}{2} \geq 0$

$\Rightarrow u + v \leq 3, \quad u \geq 1, \quad v \geq 0$

Exb. $f(z) = \frac{z+b}{z}, \quad |b| < 1, \quad b \in \mathbb{R}$

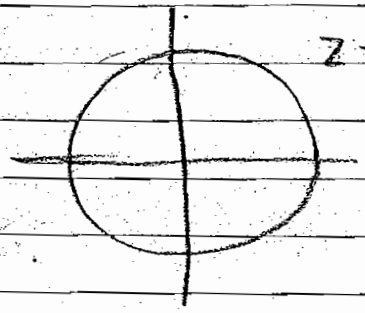
$z = e^{i\theta}, \quad \theta \in [0, 2\pi]$

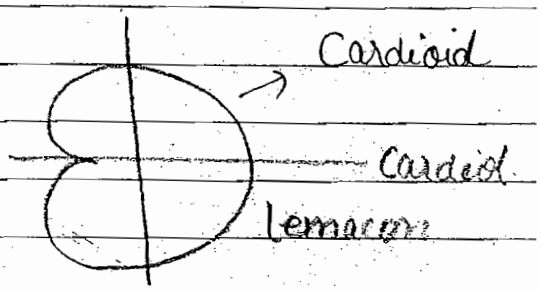




$$w = e^{i\theta} + be^{-i\theta}$$
$$= (1+b)\cos\theta + (1-b)\, i \sin\theta$$
$$u = (1+b)\cos\theta, \quad v = (1-b)\sin\theta$$
$$\frac{u^2}{(1+b)^2} + \frac{v^2}{(1-b)^2} = 1$$
$$\Rightarrow \frac{u^2}{\alpha^2} + \frac{v^2}{\beta^2} = 1$$

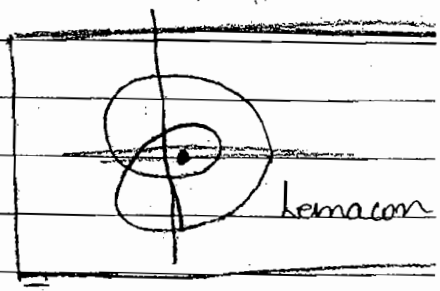
Exp: $w = z^2 + 2z$.



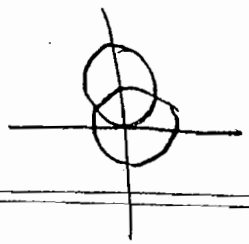


$r = a + b\cos\theta$, $b > a$ Lemacon
$r = a + b\cos\theta$, $b \leq a$, Cardioid

Ex:- Let $w = (z-1)^2(z-2)^3$
C; $|z - z_0| = r$
where $z = z_0 + re^{i\theta}$
$\theta \in [0, 2\pi]$



Sol^n:- $w = (z-1)^2(z-2)^3$
$$\arg w = 2\arg(z-1) + 3\arg(z-2)$$
$$= 2\arg w_1 + 3\arg w_2$$
$w_1 = z - 1$, $w_2 = z - 2$
$w_1 + 1 - z_0 = z - z_0$, $w_2 + 2 - z_0 = z - z_0$
$|w_1 + 1 - z_0| = r$, $|w_2 + 2 - z_0| = r$
i.e. $w_1 : |w_1 - \alpha| = r$, $\alpha = z_0 - 1$
$w_2 : |w_2 - \beta| = r$, $\beta = z_0 - 2$



if $z_0 = 0$

$|w_1 - (-1)| = r$

$|w_2 - (-2)| = r$

z can not be -1 & -2.

if Radius < 1 then

change in arg $= 0$ 0 Zeros

if $r \leq 2$ then Net arg $= 2 \times 2\pi = 4\pi$ 2

if $r > 2$, then Net arg $= 4\pi + 6\pi = 10\pi$ 5

Q: $|z - i^\circ| = \frac{1}{2}$ (Not fun^c)

$|z| = 2$, $|z| = 3$

$w = \sqrt{(z^2+1)}$ find Net argu.

$$\arg w = \frac{1}{2} \arg (z^2+1)$$

$$= \frac{1}{2} \arg (z+i^\circ) + \frac{1}{2} \arg (z-i^\circ)$$

$$= \frac{1}{2} \arg w_1 + \frac{1}{2} \arg w_2$$

$w_1 = (z+i^\circ)$, $w_2 = z-i^\circ$

$w_1 + i^\circ - z_0 = z - z_0$, $w_2 + i^\circ - z_0 = z - z_0$

$\Rightarrow |w_1 - i^\circ - z_0| = r$, $|w_2 + i - z_0| = r$

$\Rightarrow |w_1 - \alpha| = r$, $\alpha = (z_0 + i^\circ)$

$\Rightarrow |w_2 - \beta| = r$, $\beta = (z_0 - i^\circ)$

$|z| = 2$, Radius $= 2$, $w = \sqrt{z^2+1}$

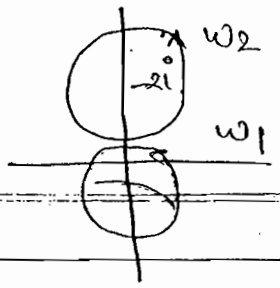
$z_0 = 0$

$|w_1 - i^\circ| = r$, z can not be i°

$|w_2 + i^\circ| = r$

$|w_1| = 1/2$

$|w_2 - 2i^\circ| = 1/2$



Change in arg w =

$$\frac{1}{2} \times 2\pi + 0 = \pi \neq 2\pi$$

$(0,-1)$

Sol. arg $w = \frac{1}{2} \times 2\pi + \frac{1}{2} 2\pi = 2\pi$

Multi-valued function :-

A relation $w = f(z)$ is defined as an MVF if $\exists$ a pair of pts z_1, z_2 s.t.

$$f(z_1) \neq f(z_2)$$

$z_1 = re^{i\alpha}$, $z_2 = re^{i\alpha + 2\pi}$

But $f(z_1) \neq f(z_2)$

Exp. $w = \sqrt{z}$

$z_1 = re^{i\alpha}$

$z_2 = re^{i(\alpha+2\pi)}$

$f(z_1) = \sqrt{r}e^{i\alpha/2}$

$f(z_2) = \sqrt{r}e^{i\left(\frac{\alpha+2\pi}{2}\right)}$

$f(z_1) \neq f(z_2)$

$\therefore$ It is not a fun^c

It is Multivalued fun^c.

$w = \sqrt{z}$ $\quad (\arg z = 2\pi,\ \arg w = \pi)$

$\arg w = \frac{1}{2} \arg z$

after n- revolution it equal $\Rightarrow$ n-valued fun^c

It is 2-valued fun^c.

$z^{1/3} \rightarrow$ 3-valued fun^c

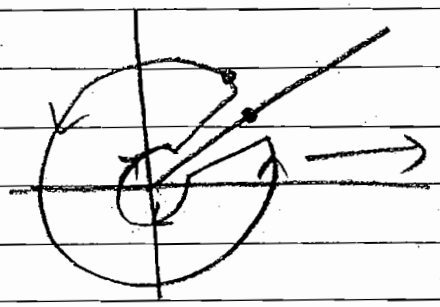
* **Branch Point** :→ Let $w = f(z)$ be a MVF then z_0 is said to be a branch point if $\exists$ $\delta > 0$ s.t. whenever pair of pts z_1, z_2 are obtain by single revolution on $|z - z_0| = r < \delta$

Δ $f(z_1) \neq f(z_2)$ Branch point

for exps. → (1) $w = \sqrt{z}$ 0

(2) $w = \sqrt{z^2+1}$ $i, -i$

(3) $w = (z-1)^{1/3}(z-i)^{1/2} z^{1/3}$ $1, i, 0$



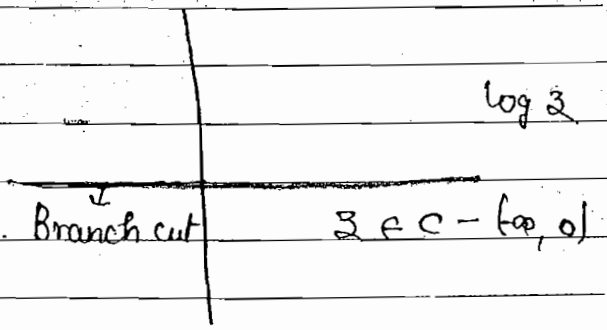
Any line segment emerging from Branch point is defined as Branch cut corresponding to that branch point provided in any movement z never crosses this line

$z_0 = x_0 + iy_0$ B.P.

$y - y_0 = m(x - x_0)$

Exp. $\log z$ is M.V.F.

It is called principal Branch.



Branch point –

$w = \sqrt{z}$ $z = 0$

$w = \log_e z$ $z = 0$

$w = \sin^{-1} z, \cos^{-1} z$ $z = \pm 1$

$w = \tan^{-1} z$ $z = \pm 1$

$w = z^z$ $z = 0$

$(z-1) + \frac{1}{2}(z-2)^{1/2} + \frac{1}{3}(z-3)$

$2\pi + \frac{1}{2}\times 2\pi + \frac{1}{3}\times 2\pi =$

not Bran — B.P. — B.P. — $(n-1)$

1. $W = \prod_{r=1}^{n} (z-r)^{1/r}$ — $n-1$ — $r=1$ integer $\therefore n-1$

2. $W = \prod_{r=1}^{n} (z-r)^{n/r}$ — $n - \tau(n)$ — $r = \tau(n)$ Integer

3. $W = \prod_{r=1}^{n} (z-1)^{1/r}$ — 1

Sol. (1) $\arg w = \frac{1}{r} \sum \arg(z-r)$

$\sum_{r=1}^{n} (z-r)^{1/2} = (z-1)(z-2)^{1/2} \cdots (z-n)^{1/n}$

$\arg w = \arg(z-1) + \frac{1}{2}\arg(z-2) \cdots$

$|w_1| = |z-1| = r$

$w_2 = z-2$

$|w_2+1| = |z-1|$

S/L

Exp. $w = z^{1/3}$ — if $(a,b) \neq (a,c) \in S \Rightarrow b = c$

$(1,1)$

$(1,\omega)$ — $\therefore 1 \neq \omega, \ \omega \neq \omega^2$

$(1,\omega^2)$

w is not a func^c it is multivalued func^c

$w = az + b$ (linear polynomial)

$|z - z_0| = r$

$\Rightarrow z - z_0 = re^{i\theta}$

$\frac{1}{a}(w-b) = z$

$\Rightarrow \left|\frac{1}{a}(w-b) - z_0\right| = |z - z_0| = r$

$\Rightarrow |w - b - az_0| = |a| \cdot r$

$\text{Arg}\left(\frac{1}{2}\right) = \{2n\pi : n \in \mathbb{Z}\}$ $\quad \arg\left(\frac{1}{2}\right) = 0$

$\arg\left(-\frac{1}{2}\right) = \{2n\pi + \pi : n \in \mathbb{Z}\}$

$|w - \alpha| = r'$ where $\alpha = b + qz_0$

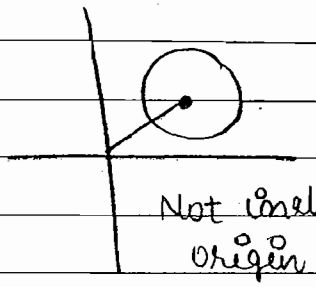
$r' = |a| \cdot r$

if z moves in circle then w is also moves in circle plane.

If radius of $z = -b/a$

then w moves around circle. if centre $= -b/a$

Change in argument of $w = 2\pi$ ($\alpha = 0$.



Ex. $w = (z-2)^{1/3}$

$\arg w = \frac{1}{3} \arg(z-2)$

$w_1 = z - 2$

$w_1 = z_0 + 2 = z - z_0$

$|w_1 - (z_0 - 2)| = |z - z_0| = r$

$|w_1 - \alpha| = r$

$\alpha = z_0 + 2 \quad < |\alpha|$

Origin outside the circle.

i.e. if radius $> |z_0 - 2|$

then change in argument $= 2\pi$

if radius $< |z_0 - 2|$ then change in argument $= 0$

if $z_0 = 2$, $\alpha = 0$

Minimum no. of revolution = order of Branch cut

$\arg(w) = \frac{1}{3} \times 2\pi$

$\arg(z) = 2\pi$

$\arg(w) \neq \arg(z)$ at $z_0 = 2$

$\therefore$ 2 is Branch point.

No. of revolution $= 1/3$.

3rd UNIT (LCD)

Let $f: D \subseteq \mathbb{C} \to \mathbb{C}$

Then we can write $w = f(z)$

$$f(z) = f(x+iy) = u(x+iy) + iv(x+iy)$$
$$= u(x,y) + iv(x,y)$$

1) Now if $S = \{(x,y) \in \mathbb{R}^2 : x+iy \in D\}$

Then u & v are real valued fun^c defined on S.

2) If $z_0 = x_0 + iy_0 \in D'$

$\Leftrightarrow (x_0, y_0) \in S'$

Eg:-
$$f(z) = \frac{1}{z} = \frac{|\bar{z}|}{|z|^2}$$

where $\bar{z} = x - iy$ is called 'a' conjugate of z

$$f(z) = \frac{x}{(x^2+y^2)} + \frac{(-y)}{(x^2+y^2)} i$$

$$u = \frac{x}{x^2+y^2}, \qquad v = \frac{-y}{x^2+y^2}$$

'OR'
$$f(z) = \bar{z} = x - iy$$
$$u(x,y) = x; \quad v(x,y) = -y$$

LIMIT :→ Let $z_0 \in D'$ and $l = l_1 + il_2 \in \mathbb{C}$

we say $\lim_{z \to z_0} f(z) = l$

⇒ for any $\varepsilon > 0$ $\exists$ $\delta > 0$

s.t. $|f(z) - l| < \varepsilon$

whenever $|z - z_0| < \delta$ in $D - \{z_0\}$

Result :→ $\lim_{z \to z_0} f(z)$ at exists

iff that of $u(x,y)$ & $v(x,y)$ exists at (x_0, y_0)

& $\lim_{(x,y) \to (x_0,y_0)} u(x,y) = l_1$

& $\lim_{(x,y) \to (x_0,y_0)} v(x,y) = l_2$

Then $\boxed{\lim_{z \to z_0} f(z) = l_1 + i l_2}$

Exp. $f(z) = \bar{z}$

at every point limit exist.

Let $f : \mathbb{N} \longrightarrow \mathbb{C}$

then $f(n) = z_n$

defines a sequence of complex no. $\langle z_n \rangle$

& $z_n = a_n + i b_n$

where $\langle a_n \rangle$ & $\langle b_n \rangle$ are seq. of real no.

* $\lim_{z \to z_0} f(z) = l$

if $\langle z_n \rangle \to z_0$

$\langle f(z_n) \rangle \longrightarrow l$

Exp. $f(z) = e^{-1/z^2}$

$a_n = \frac{1}{n} \to 0 \quad f(a_n) = e^{-n^2} \longrightarrow 0 \Longrightarrow$

$b_n = \frac{i}{n} \to 0, \quad f(b_n) = e^{n^2} \to \infty$

$c_n = \frac{1}{i\sqrt{2n\pi}} \to 0, \quad f(c_n) \longrightarrow 1$

e^{-1/z^2} → Limit Does not exist at 0

e^{-1/x^2} → Limit exist.

Limit Does not exist at 0.

Exp. $f(z) = \dfrac{\bar{z}}{z}$, $z \neq 0$

$$f(z) = \frac{x - iy}{x + iy}$$

along $y = mx$, $\quad f(x) = \dfrac{x - imx}{x + imx}$

$$\longrightarrow \frac{1 - im}{1 + im}$$ Limit Does not exist.

Exp. $A_0 = \{ \langle z_n \rangle : \lim z_n = 0 \}$

$B_0 = \{ \lim f(z_n)$ if exists : $\langle z_n \rangle \in A_0 \}$

$$\boxed{B_0 = \phi}$$

$f : D \to \mathbb{C}$; $D \subseteq \mathbb{C}$

$w = f(z) = u(x, y) + iv(x, y)$

$\lim_{z \to z_0} f(z) = f(z_0)$

$=$

$f(z) = x - iy$

u & v are conti. at (x_0, y_0)

$\Leftrightarrow$ f is conti. of $z = x_0 + iy_0$

NOTE :- $f : D \to \mathbb{C}$

f is said to be bounded if

$\exists \; M \in \mathbb{R}$

s.t. $|f(z)| \leq M$

$\forall \; z \in D$

- f is unbdd if for any $n \in \mathbb{N}$ $\exists\ z_n \in D$ s.t. $|f(z_n)| > n$

& $f(z) \neq \infty$

Result:- If f is conti. at z_0 or Limit exist finitely then in a nbd of z_0 f is bdd. =

Note :- $$f(z) = \begin{cases} 1/z & , z \neq 0 \\ \infty & , z = 0 \end{cases}$$

Then f is continuous at $z = 0$

$$f(z) = \begin{cases} 1/z & , z \neq 0 \\ \infty & , z = 0 \\ 0 & , z = \infty \end{cases}$$

f is continuous in C_∞.

Exp. $f(z) = e^z$

$= e^{x+iy} \Rightarrow e^x \cdot e^{iy}$

$\Rightarrow e^x(\cos x + i \sin y)$

$= \underbrace{e^x \cos x}_{u} + i \underbrace{e^x \sin y}_{v}$

Exp. $f(z) = \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin iy$

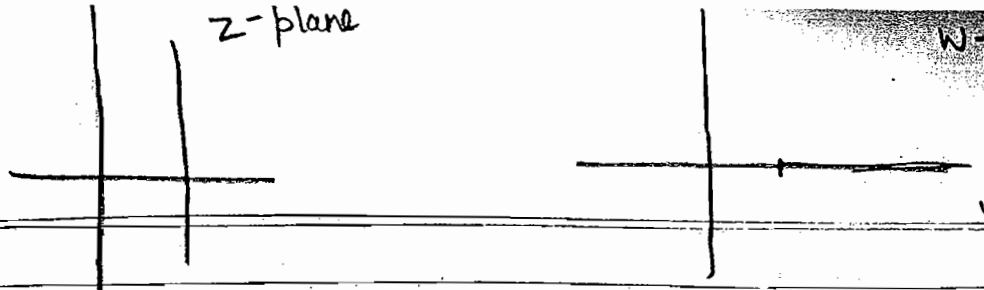
$= \sin x \cosh y + i \cos x \sinh y$

$$\cos iy = \cosh y = \frac{e^x + e^{-x}}{2}$$

$$\sin iy = i \sinh y = i\left(\frac{e^x - e^{-x}}{2}\right)$$

where $x = \pi/2$,

$f(z) = \cosh y = \frac{e^x + e^{-x}}{2}$



Hence $\sin z$ is not bdd

" $\cos z$ is not bdd. But $\sin x$ & $\cos x$ are bdd in $\mathbb{R}$.

DIFFERENTIABILITY :- Let $D \subset \mathbb{C}$

$|z - z_0| < \delta \subset D$ for some $\delta > 0$

Let $f : D \longrightarrow \mathbb{C}$

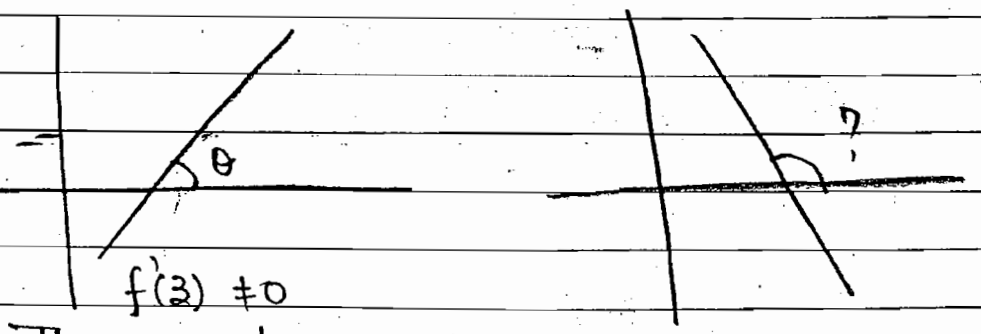
Define $\phi : D - \{z_0\} \longrightarrow \mathbb{C}$

as $\phi(z) = \dfrac{f(z) - f(z_0)}{z - z_0}$

Then f is said to be diff. at z_0 iff $\lim_{z \to z_0} \phi(z)$ exist finitely.

then $\boxed{\lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0} = f'(z_0)}$

and called the derivative of f at z_0.



Then $\theta = \phi$

if $f'(z) = 0$ then root of zeros k

then $\phi = (k+1)\theta$

eg:- $f'(z) = 3z^2$ $(2+1)$

then $\phi = 3\theta$ //.

Exp. $f(z) = |z|^2$

$z_0 = 0$ check diff. ?

Sol. $\lim_{z \to 0} \frac{|z|^2 - |z|^2}{z - 0} = 0$

$f(z) = z\bar{z}$

$\phi(z) = \frac{f(z) - f(0)}{z - 0}$

$= \frac{z\bar{z} - 0}{z} = \bar{z}$

$\phi(z) = \bar{z}$

It is diff.

Exp. $f(z) = \bar{z}$

$\phi(z) = \frac{\bar{z} - 0}{z - 0} = \frac{\bar{z}}{z}$

It is not diff.

N^{y} Condition for Differentiability :→

1) If f is diff. at z_0 then at ordered pair (x_0, y_0) u & v possess partial derivative with respect to x & y both.

i.e. u_x, u_y, v_x, v_y exist & satisfied.

$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\}$$

Called C-R Equ^{n}.

Exp $f(z) = |z|^2 = (x^2 + y^2) + 0 \cdot i$

$u = x^2 + y^2, \quad v = 0$

$u_x = 2x, \quad v_x = 0$

$u_y = 2y, \quad v_y = 0$

$P \rightarrow q$ (N.Y)

$u_x = v_y = x = 0$

$y = 0$

$f(z) = 0 + 0i = 0$

i.e. 0 is only point where $f(z)$ diff.

Exp. $f(z) = \bar{z} = x - iy$

$u_x = 1 \qquad v_x = 0$

$u_y = 0 \qquad v_y = -1$

$u_x = v_y$

$1 \neq -1$ So $f(z)$ is not diff.

it is throughout continuous nowhere diff.

Sufficient Condition :→ If at (x_0, y_0) u_x, u_y, v_x, v_y exist, continuous and satisfy C-R Eqn then f is differentiable at $z_0 = x_0 + iy_0$

& $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$

$= v_y(x_0, y_0) - iu_y(x_0, y_0)$

$= -i[u_y(x_0, y_0) + iv_y(x_0, y_0)]$

$f'(z) = u_x + iv_x$

$= -i(u_y + iv_y)$

Exp. $f(z) = x^2 + y^2 + 2xyi$

$u_x = 2x, \quad u_y = 2y$

$v_x = 2y, \quad v_y$

$u_x = 2x, \quad v_y = 2x$

$u_y = 2y, \quad v_x = 2y$

It is diff. at X-axis,

Set of pts where f is diff. are uncountable.

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. $f(z) = p(z) = \sum_{i=0}^{n} a_i z^i$

. $f(z) = \sin z$

3. $f(z) = \cos z$

4. $f(z) = e^z$

5. $f(z) = \frac{f_i(z)}{e^z}$ $\quad f_i = 1$ to 5

6. $f(z) = \alpha f_i(\beta z) + \beta f_j(z)$, f_i & $f_j = 1$ to 6.

7. $f(z) = f_i(z) \cdot g_j(z)$ $\quad f_i$ & $f_j = 1$ to 7

Exp. 1 - 7 are diff $\forall\ z \in \mathbb{C}$

8. $h(z) = \frac{f(z)}{g(z)}$ $\quad f$ & g are 1 to 7

h is diff on $\mathbb{C} - N$

$N = \{ z \in \mathbb{C} : g(z) = 0 \}$

9. $h(z) = f \circ g(z) = f[g(z)]$ $\quad f: 1$ to 7

$g: 0$

Exp. $\cos \cdot \frac{1}{z}$

$(\sin 1/z)$

$\sin \frac{1}{z} = 0 \Rightarrow \frac{1}{z} = n\pi$

$z = 1/n\pi$ Not def.

& $\frac{1}{z} = 0 \Rightarrow \sin 0 = 0$

$\cos\left(\frac{1}{0}\right) \rightarrow$ Not def.

DOD $= \mathbb{C} - N$

where $N = \{0\} \cup \left\{ \frac{1}{n\pi} : n \in \mathbb{Z} - \{0\} \right\}$

S	A
T	C

Q. $f(z) = \frac{1}{\sin z - \cos z}$

$\sin z - \cos z = 0$

$\tan z = 1$

$z = n\pi + \pi/4$

$f(z)$ is diff. on $\mathbb{C} - N$

where $N = \{ n\pi + \frac{\pi}{4} : n \in \mathbb{Z} \}$

Q. $f(z) = \frac{1}{\sin \pi z}$

$\sin \pi z = 0 \Rightarrow \pi z = n\pi$

$N = \mathbb{Z}$

$f(z)$ is diff on $\mathbb{C} - \mathbb{Z}$

* Regular Point : Let f is defined on D and $z_0 \in D$

we say z_0 is regular point of f or equivalently we say f is said to be regular at z_0.

if $\exists\ \delta > 0$ s.t. $f'(z)$ exists $\forall\ z \in |z - z_0| < \delta$

Result :- 1) If f is regular at z_0 then $\exists\ \delta > 0$ s.t. $f'(z)$ exists $\forall\ z \in |z - z_0| < \delta$

Let $t \in |z - z_0| < \delta$

as $|z - z_0| < \delta$ is open $\Rightarrow \exists\ \delta'$ s.t.

$|z - t| < \delta' \subset |z - z_0| < \delta$

$\Rightarrow f'(z)$ exists $\forall\ z \in |z - t| < \delta'$

$\Rightarrow t$ is regular point.

i.e. all the elements of $|z - z_0| < \delta$ are the regular points of f

There are infinite regular points.

) If R = The set of regular points of f then
$R \subseteq D$ (Dod)
& R is open as if $z_0 \in R$ —
$\exists \ \delta > 0$ s.t. $|z - z_0| < \delta \subset R$ by ①

In exp →
(1) → (7) Regular point = ℂ

;) $R \subseteq R'$

Singular point :→ " $z_0 \in C$ is said to be a singular point of f if it is a l.p. of the set of regular point but not regular itself.
i.e. define $S = R' - R$
S is set of Singular point.
If z_0 is singular point of f, we say f has singularity at z_0.

$w = f(z)$

D
R $R \subseteq D$
R' $R \subseteq R'$
S $= R' - R$

eg.- $f(z) = \sin \frac{1}{z}$

$D = \mathbb{C} - \{0\}$
$R = D$
$R' = C$
$S = R' - R = \{0\}$

Let $\alpha \in S' \Rightarrow \alpha \in R'$

$|z-\alpha| < \delta \cap S-\{\alpha\}$ is Infinite

Let $t \in |z-\alpha| < \delta \cap S-\{\alpha\}$

$\exists\ \delta'$ s.t. $|z-t| < \delta \cap R$ = Infinite

$\Rightarrow t \in S$

$\because \exists\ \delta'$, $|z-t| < \delta' \subset |z-\alpha| < \delta$

$\therefore |z-t| < \delta' \cap R$ is Infinite.

i.e. $\alpha \in S'$

$\Rightarrow \alpha \in R'$

if $\alpha \in R$ $\exists\ \delta > 0$ s.t.

$|z-\alpha| < \delta \subset R$

$\Rightarrow |z-\alpha| < \delta \cap S = \phi$

$\Rightarrow \alpha \notin S'$ ✗.

Hence, if $\alpha \in S' \Rightarrow \alpha \in R'$ & $\alpha \notin R$

$\Rightarrow \alpha \in R'-R = S$

$\boxed{S' \subseteq S}$

Row Set of singular points is closed.

Exp. $f(z) = |z|^2$

$D = \mathbb{C}$

$R = \phi$, $R' = \phi$

$S = \phi$

It has no singularity.

Exp. $f(z) = \dfrac{1}{\sin \pi z}$

$S = \mathbb{Z}$

$S' = \{\infty\}$ in $\mathbb{C}_\infty$

$= \phi$ in $\mathbb{C}$

Exp. $f(z) = \frac{1}{\sin 1/z}$

$\sin \frac{1}{z} = 0 \Rightarrow z = \frac{1}{n\pi}$.

$S' = \{0\}$ (limit pt of S)

Exp. $f(z) = \begin{cases} 1 & |z| < 1 \\ 2 & |z| \geq 1 \end{cases}$

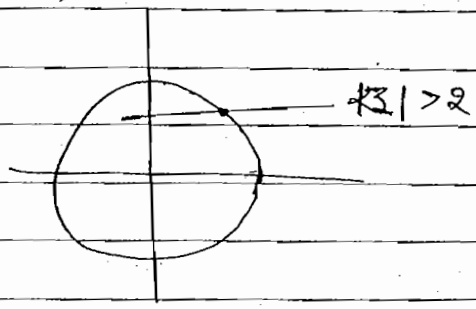
$D = \mathbb{C}$

$R = \mathbb{C} - \{z \in \mathbb{C} : |z| = 1\}$

$R' = \mathbb{C}$

$S = \{z \in \mathbb{C} : |z| = 1\}$

$S' = S$



Exp. $f(z) = \log z$

$D = \mathbb{C} - (-\infty, 0]$

$R = D$

$R' = \mathbb{C}$

$S = (-\infty, 0]$

$S' = S$

Classification of Singularities :→

1) Point Set & Classification

(a) Isolated Singularities :- $z_0 \in S$ is called isolated Singularities if $\exists\ \delta > 0$ s.t.

$0 < |z - z_0| < \delta \subset R$

i.e. f is regular at every point of $|z|$

$0 < |z - z_0| < \delta$

i.e. $z_0 \in S - S'$

Analytic :-

"f is Said to be analytic on D if it is regular of every point of D."

$\Rightarrow$

2) **Non-Isolated :-** $z_0 \in S$ is Said to be non-isolated singularity of f if it is limit pt of S. i.e. in every nbd of z_0 there is atleast one singularities of f other than z_0.

$S' \rightarrow$ Set of Non Isolated

$I = S - S' \rightarrow$ Set of Isolated.

Exp $\quad f(z) = \sin \frac{1}{z}$.

$z_0 = 0$, only one singularities

It is Isolated singularity.

* S $\rightarrow$ finite every sing. has to be isolated.
* If fun^c has non-Isolated singu. i.e. S is Infinite

Exp. $\quad f(z) = \frac{1}{\sin \pi z}$

$S = \mathbb{Z}, \quad S' = \{\infty\}$

f is Non-Isolated Singularity

Note :- f_∞ is non-isolated sing. if S is unbdd.

NOTE :- The behaviour of f at ∞ is analysis by $g(z) = f(1/z)$ at $z = 0$

Exp. $\quad f(z) = \sin z$

$g(z) = \sin 1/z$

at 0, ∞ is Sing.

Exp. $f(z) = \frac{1}{\sin \pi z}$ Discuss Sing.

$S = \mathbb{Z}$ in $\mathbb{C}$.

$S' = \{\infty\}$, $\mathbb{C}_\infty$

$g(z) = \frac{1}{\sin \pi/z}$

$g(z)$ it has singularity at 0

$\therefore f(z)$ has singul. at ∞.

$\hookrightarrow$ it @ is Non - Isolated Sing.

Exp. $f(z) = e^{\frac{1}{z}}$

$g(z) = e^{1/z}$

$g(z)$ has isolated sing. at 0

$\therefore f(z)$ has non isolated singularity at ∞.

Exp. $f(z) = \frac{1}{e^z - 1}$

$S = \{2n\pi i \; ; \; n \in \mathbb{Z}\}$ in $\mathbb{C}$

$g(z) = \frac{1}{e^{1/z} - 1}$ check at $z = 0$

$= \frac{1}{\infty - 1} = 0$ $\left(z = \frac{1}{2n\pi i^\circ}\right)$

$\therefore g$ has non - Isolated sing. at 0

f has " " " at ∞.

Exp. $f(z) = \frac{1}{z}$

$z = 0$ is Singul. $\therefore$ Isolated

but $f(z)$ has no sing. at ∞.

$$f(x) = \frac{1}{z-1} + \frac{2}{2z-1}$$

$S = \{1, 1/2\} \longrightarrow$ Isolated

$S' = \phi$

S is bdd.

$$g(z) = \frac{z}{1-z} + \frac{2z}{2-z}$$

at 0 it has no-sing.

$\therefore f(z)$ has no-sing. at ∞ $\quad (0 \notin \{1,2\})$

$\therefore S = \{1, 2\}$

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$$f(z) = \frac{1}{\sin \pi z} + \frac{1}{\sin 1/z}$$

$$S = \{0\} \cup \mathbb{Z}' \cup \left\{ \frac{1}{n\pi} ; n \in \mathbb{Z} - \{0\} \right\}$$

It is unbdd fun^c, it has atleast one limit pt in $\mathbb{C}_\infty$.

$S' = \{0, \infty\}$ in $\mathbb{C}_\infty$

It is Non-Isolated Singularity.

★ Classification of Isolated Singularities on the basis of character of f :⟶

Let $z_0 \in S - S'$ ☛

⟹

1) Removable Sing. :⟶ $\lim_{z \to z_0} f(z)$ exists finitely.

Eg ⟶ $f(z) = \frac{\sin z}{z}$, $z \neq 0$

$D = \mathbb{C} - \{0\}$

$R = \mathbb{C} - \{0\}$, $R' = \mathbb{C}$, $S = \{0\}$

$\frac{1-\cos z}{2z\sin z + z^2\cos z}$ $\frac{0}{0}$

$c \{0\}$ → check cond. at 0. $\frac{\cos z}{1} \Rightarrow \frac{1}{1}$

2. $f(z) = \frac{\sin z}{z}$

$D = \mathbb{C} - \{0\}$

$R = \mathbb{C} - \{0\}$, $R' = \phi$

$S = \{0\}$

This fun^c has limit at 0.

3. $f(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 0 & z = 0 \end{cases}$

$D = \phi$

$R = D - \{0\}$

$R' = \mathbb{C}$

$S = \{0\}$

(at 0 limit does not exist but $0 \neq 1$ So not continuous not diff.)

This fun^c has limit at 0.

4. $f(z) = \frac{z - \sin z}{z^2 \sin z}$

$D = \mathbb{C} - \{n\pi : n \in \mathbb{Z}\}$

$R = D$ $R' = \phi$

$S = \{n\pi : n \in \mathbb{Z}\}$

at $z = 0$ limit exist

For limit exist - diff. on Den. & Num. and put value for limit if num = Deno then limit exist

In all exmple's 0 is removable singularity.

NOTE:- Near removable sing. fun^c is bdd.

$\exists n \in |z - z_0| < 1/n$

$|f(z_n)| > n$

2) Pole Singularity :→ $\lim_{z \to z_0} f(z)$ does not exist.

But $\exists\ m \in \mathbb{N}$ s.t.

$\lim_{z \to z_0} (z - z_0)^m f(z) = l$ finitely & Non zero

Then z_0 is called pole of order m.

m is unique.

Let $\langle a_n \rangle \to z_0$

$\langle f(a_n) \rangle \longrightarrow l$ (finite)

Then $\lim_{n \to \infty} (a_n - z_0)^m f(a_n) = 0$

i.e. $\boxed{\lim_{z \to z_0} f(z) = \infty}$

fun^c is unbdd in a nbd of pole

Note :- $z = z_0$ is pole iff $\lim_{z \to z_0} f(z) = \infty$

Exp. $f(z) = \frac{1}{z}$

$\lim_{z \to 0} \frac{1}{z} \to \infty$ ∴ pole Sing.

$f(z) = \frac{1}{z} \cdot z \Rightarrow \lim_{z \to 0} \frac{1}{z} \cdot z =$ Simple pole.

$f(z) = \frac{\sin z}{z^3}$

$\lim_{z \to 0} \frac{1}{z^2} \cdot \frac{\sin z}{z} \Rightarrow \lim_{z \to 0} z^2 \cdot \frac{1}{z^2} \frac{\sin z}{z} = 1$

2 order pole.

$\frac{f(z)}{g(z)} = \frac{z - \sin z}{z^2 \sin z^2}$, $f'(z)=0$, $f''(z)=0$, $f'''(z) \neq 0$

$g'(0)=0$, $g''(0)=0$, $g'''(0)=0$, $g''''(0) \neq 0$

$\frac{\text{Order of } f(z)}{\text{Order of } g(z)} = \frac{3}{4}$ at $z=0$

$\therefore$ Order of Sing. $= 4-3 = 1$

if $f(z) \geq g(z) \rightarrow$ Removable

$f(z) < g(z) \rightarrow$ Pole.

$g(z) - f(z) \rightarrow$ order of pole.

3) Essential Singularity :→ A singularity neither branch pt nor removable nor pole are called Essential Singularity.

NOTE :- Non-Isolated sing. are also refered as Essential Sing.

Exp. $f(z) = e^{1/z}$

$a_n = 1/n \rightarrow$ pole

$\therefore f(z) \rightarrow \infty$

$b_n = -1/n \rightarrow f(z) \rightarrow 0$

$c_n = 1/2n\pi i, \; f(z) \rightarrow 1$

at b_n, c_n $f(z)$ has essential sing.

Exp. $f(z) = \frac{1}{\sin \pi/z}$

$S = \left(\frac{1}{n} : n \in z - \{0\} \right) \cup \{0\}$

'0' is non-Isolated essential Sing.

Exp. $f(z) = z^2 \sin 1/z$

$\lim_{z \to \infty} f(z) = 0 \longrightarrow$ Not pole.

It is Isolated Essential Sing.

Exp. $\cos \frac{1}{\sin \frac{1}{z}}$ at $z = 0$ f has.

a) Isolated

b) Non - Isolated

c) Essential

d) pole.

But (c) also true

$\because$ Non isolated $\longrightarrow$ Essential.

Exp. 1) $f(z) = \frac{1}{\sin \pi/z}$

2) $f(z) = \begin{cases} 1, & |z| < 1 \\ 2, & |z| \geq 1 \end{cases}$

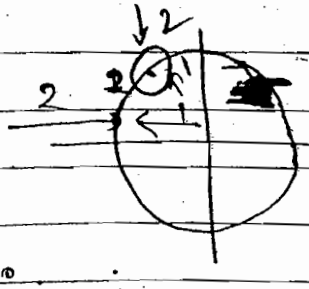
Sol(1) $\Rightarrow \sin \frac{\pi}{z} = 0, \quad \frac{\pi}{z} = n\pi \Rightarrow z = \frac{1}{n}$

$\left\{ \frac{1}{n} : n \in \mathbb{Z} \right\} \longrightarrow$ Non Isolated

$\therefore$ Essential.

(2) $|z| = 1$ are Singular point which are infinite.

$\therefore$ Essential Singularity.



Limit does not exist

NOTE :- (1) In a nbd of Non-Isolated singularity which is essential always, fun^c may be bdd or unbdd.

In a nbd of isolated essential singularity, fun^c are unbdd. (by C-W Theorm)

Proof :- If z_0 is isolated essential sing.

$\forall\ c \in \mathbb{C}\ \exists\ \langle z_n \rangle \to z_0$

$\langle f(z_n) \rangle \to c$

$A_{z_0} = \{ \langle z_n \rangle : \langle z_n \rangle \to z_0 \}$

$B_{z_0} = \{ \lim_{n\to\infty} f(z_n) \text{ if exists} : \langle z_n \rangle \in A_{z_0} \}$

Let z_0 is an isolated sing. of f

R.S. $B_{z_0} = \{l\}$

Pole $B_{z_0} = \{\infty\}$

Essential $B_{z_0} = \mathbb{C}_\infty$

when $l = \lim_{z\to z_0} f(z)$

Properties of Analytic fun^c :→

$f : D \to C$

D is open & connected = Domain

$T = \{ (x,y) : x+iy \in D \}$

then in $w = f(z) = u(x,y) + iv(x,y)$

$u : T \to \mathbb{R}$

$v : T \to \mathbb{R}$

D = X ∪ Y

X & Y are open & disjoint

⇒ X = ϕ or Y = ϕ

$f'(z) = v_y - iu_y$

$= i(u_y + iv_y)$

Now Let f is analytic in D

where $f(z) = u(x,y) + iv(x,y)$

[C1]: $f'(z) = 0 \quad z \in D$

f is constant

[C2] In any of the following condition f is constant

(a) Any of u or v is constant.

(b) $|f(z)|$ is constant

Proof:-

$$u^2 + v^2 = c^2$$

$$uu_x + vv_y = 0$$

$$uu_y + vv_y = 0$$

$$\Rightarrow \quad uu_x - vu_y = 0 \quad \Rightarrow \quad \begin{bmatrix} u & -v \\ v & u \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$vu_x + uu_y = 0$$

$$AX = 0$$

$$|A| = u^2 + v^2 = c^2 \neq 0$$

$$u_x = 0, \quad u_y = 0$$

(c) Arg f is constant.

$$\text{Arg } f = \tan^{-1}\left|\frac{y}{x}\right| = \lambda$$

$$= \tan^{-1}\frac{v}{u} = \lambda \Rightarrow \quad u = v\tan\lambda$$

$$u\tan\lambda - v = 0$$

$$g(z) = (\tan\lambda + i^\circ)\cdot f(z)$$

$$= (\tan\lambda + i^\circ)(u + i^\circ v)$$

$$= (u\tan\lambda - v) + i^\circ(v\tan\lambda + u)$$

$$= U + i^\circ V$$

$$= 0 + i^\circ V$$

$$g(z) = i^\circ c \quad \text{by (a)}$$

$$\boxed{f(z) = \frac{\tan\lambda + i^\circ}{i^\circ c}}$$

(d) $au + bv = c$

$g(z) = (a - ib) \cdot (u + iv)$

$g(z) = (au + bv) + i(av - bu)$

$= U + iV$

$= c + iV$

$g(z)$ is constant $\quad g(z) = c + id$

$$\boxed{f(z) = \frac{c + id}{a - ib}}$$

e) $u = v^2$ $\qquad$ 10 | 11 | 2014

6] $f(z) = u(x, y) + iv(x, y)$

1. u & v are harmonic

i.e. $\nabla^2 u = 0$ & $\nabla^2 v = 0$

$u_x = v_y, \quad u_y = -v_x$

$u_{xx} = v_{xy}, \quad u_{yy} = -v_{yx}$

$u_{xx} + u_{yy} = v_{xy} - v_{yx}$

If continuous then $\nabla u = 0 \quad (\because v_{xy} = v_{yx})$

2. v is called Harmonic conjugate of u

i.e. Two if $\phi_1(x, y)$ & $\phi_2(x, y)$ are harmonic then ϕ_2 is called Harmonic conjugate of ϕ_1.

if $\phi_1 + i\phi_2$ is analytic.

3. If u & v are Harmonic conjugate of each other than $\Rightarrow$ both are constant.

i.e. $u + iv$ is also constant. & $(v + iu)$

* $g(z) = e^{f(z)}$

$|g(z)| = |e^{(u+iv)}| = e^u$

$\rightarrow$ const.

* $u(x,y) = \alpha$, $v(x,y) = \beta$
Represent orthogonal family of curves with α and β as constant.

for u, $du = 0$

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{u_x}{u_y} = m_1$$

for v, $\frac{dy}{dx} = -\frac{v_x}{v_y} = m_2$

$$\boxed{m_1 \cdot m_2 = -1}$$

Exp. $f(z) = z^2$

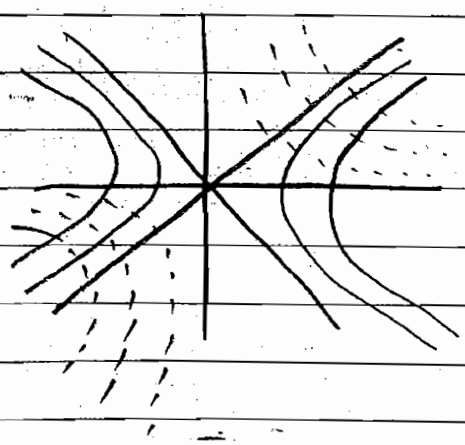
$= x^2 + y^2 + 2xiy$

$u = x^2 - y^2 = \alpha$, $v = 2xy = \beta$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$$

$h=0$, $a=-b$ → circle

$h^2 - ab > 0$, rectangle



*

$z = x + iy$

$\bar{z} = x - iy$

$\bar{z}$ is called conjugate of z.

$x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

$x = x(z, \bar{z})$, $y = y(z, \bar{z})$

$$f(z) = x^2 - y^2 + 2xyi$$

$$= \left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2 + 2i\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right)$$

$$= z^2 + \bar{z}^2 + 2z\bar{z} + z^2 + \bar{z}^2 - 2z\bar{z}$$

$$= 2z^2 + 2\bar{z}^2$$

$$f(z) = u(x,y) + iv(x,y)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}} + i\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right)$$

$$= \left(u_x \cdot \frac{1}{2} + u_y \cdot \frac{1}{2}i\right) + i\left(v_x \cdot \frac{1}{2} + \frac{1}{2}iv_y\right)$$

$$= \frac{1}{2}\left[u_x + iu_y + iv_x - v_y\right]$$

$$\boxed{\frac{\partial f}{\partial \bar{z}} = 0}$$

This is called complex form of C-R eqn.

i.e. If f is analytic in D

$\Rightarrow \frac{\partial f}{\partial \bar{z}} = 0$ on D

i.e. f is free from $\bar{z}$.

Exp. $f(z) = |z|^2$

$= z\cdot\bar{z}$

$\partial f / \partial \bar{z} = z$

at $z = 0$ $\Rightarrow$ only $\frac{\partial f}{\partial \bar{z}} = 0$

$f(z)$ is diff. at 0.

$$f(z) = x = \frac{z+\bar{z}}{2}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \quad (\text{No where } 0)$$

$\therefore$ f is No where diff.

* Formation of analytic fun^c when either real or Imaginary part given :→

(I) Let $f(z) = u + iv$ is analytic in D.

$\Rightarrow$ $\overline{f(z)} = u - iv$ is free from z.

i.e. $\overline{f(z)} = \phi(\bar{z})$ (Some fun^c of $\bar{z}$.)

Say, $\overline{f(z)} = \bar{f}(\bar{z})$, ϕ is denoted by $\bar{f}$

Now, let $u(x, y)$ is given.

We have,

$$f(x+iy) = u(x,y) + iv(x,y)$$

$$\bar{f}(x-iy) = u(x,y) - iv(x,y)$$

$$f(x+iy) = 2u(x,y) - \bar{f}(x-iy) \quad \text{---} \quad (I)$$

Put $x = \frac{z}{2}$, $y = \frac{z}{2i}$

$$\boxed{f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - \bar{f}(0)}$$

$$f(x-iy) = u(x,y) - v(x,y)$$

$$\bar{f}(0) = u(0,0) - v(0,0)$$

$$\boxed{f(z) = 2u\left(z/2, z/2\right) - u(0,0) + ci}$$ C.R.

Exp. $f(z) = x^2 - y^2$.

$u(x,y) = x^2 - y^2$, $\nabla^2 u = 0$

$$2u\left(\frac{z}{2}, \frac{z}{2i}\right) = 2\left(\frac{z^2}{4}, -\frac{z^2}{4i^2}\right)$$
$$= z^2$$

$$f(z) = \frac{2u\left(\frac{z}{2}, \frac{z}{2i}\right)}{z^2} - u(0,0) + ic$$

$$v = (2xy + c)$$

Ex: Find the most general form of cubic (homogeneous) in x & y which is harmonic as well & find $f(z)$ whose real part of this func.

Sol^n :-

$$u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$$
$$u_{xx} + u_{yy} = 0$$
$$u_x = 3ax^2 + 2bxy + cy^2$$
$$u_{xx} = 6ax + 2by$$
$$u_y = bx^2 + 2cxy + 3dy^2$$
$$u_{yy} = 2cx + 6dy$$
$$u_{xx} + u_{yy}$$
$$\Rightarrow 6ax + 2by + 2cx + 6dy = 0$$
$$\Rightarrow 2x(3a+c) + 2y(b+3d) = 0$$
$$3a + c = 0 \quad , \quad b + 3d = 0$$
$$a = -c/3 \qquad d = -b/3$$
$$f(z) \Rightarrow ax^3 + bx^2y - 3axy^2 - \frac{b}{3}y^3$$

Q:- $u = \phi(\sqrt{x^2+y^2})$

Find all the harmonic of f.

$$x^2 + y^2 = r^2$$
$$u = \phi(r)$$
$$u_{xx} + u_{yy} = 0$$

$$u_x = \phi'(r) \cdot \frac{x}{r}$$

$$u_{xx} = \phi''(r) \cdot \frac{x}{r} \cdot \frac{x}{r} + \frac{\phi'(r)}{r}$$

$$= \frac{1}{r^2} \cdot \frac{x}{r} \phi'(r) \cdot x$$

$$u_{xx} + u_{yy} = \phi''(r)\left(\frac{x^2+y^2}{r^2}\right) + \frac{2\phi'(r)}{r} - \frac{\phi'(r)}{r^3}(x^2+y^2) = 0$$

$$= \phi''(r) + \frac{2\phi'(r)}{r} - \frac{\phi'(r)}{r} = 0$$

$$= \boxed{\phi''(r) + \frac{\phi'(r)}{r} = 0}$$

(Put $\phi'(r) = \psi(r)$ eqn. is linear)

$$\Rightarrow \boxed{u(x,y) = c_1 \log(x^2+y^2) + c_2}$$

(II) $f'(z) = u_x + iv_x$

If $f'(z) = f(z)$, $z \in D$

Then we write

$$\int f(z)\,dz = F(z) + c$$

F is called anti derivative of f.

(i) $\int z^n dz = \frac{z^{n+1}}{n+1} + c$

(ii) $\int \sin z\,dz = -\cos z + c$

(iii) $\int \frac{1}{z}\,dz = \log z + c$

(iv) $\int e^z dz = e^z + c$

} are principal Branch

$$f'(z) = u_x + iv_x, \quad z \in D$$

$$= u_x - iu_y$$

$f'(x+iy) = u_x(x,y) - i u_y(x,y)$

$x = z, \quad y = 0$

$f'(z) = u_x(z,0) - i u_y(z,0)$

$= \phi_1(z) + i\phi_2(z)$

$= \psi(z)$ // MINI THOMSON METHOD

Exp. $u(x,y) = x^2 - y^2$

$u_x = 2x, \quad \phi_1(z) = 2z$

$u_y = -2y, \quad \phi_2(z) = 0$

Ans.- $2z + 0 = 2z$

I) $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$dv = -u_y dx + u_x dy$

$v = \int (-u_y dx + u_x dx) + c$

Exp. $f(z) = u(x,y) + iv(x,y)$

S.t. $au + bv = x^2 - y^2$

$g(z) = (a - bi) f(z)$

$= (a - bi)(u + iv)$

$= (au + bv) + i(av - bu)$

$= U + iV$

$\Rightarrow g(z) = z^2 + ic$

$f(z) = \frac{z^2 + ic}{a - bi}$

COMPLEX INTEGRATION

* **Curve :→** A curve γ in complex plane is defined through a map from close interval $[a,b]$ to C.

$$\gamma : [a,b] \longrightarrow C$$

$$\gamma(t) = x(t) + iy(t), \quad t \in [a,b]$$

$x(t)$ & $y(t)$ are continuous func^{c} of 't'.

Arc, conti. curve are also refered as curve.

1) If we eliminate 't' b/w $x(t)$ & $y(t)$ to that $\phi(x,y) = 0$, this ϕ is represent a curve in cartesian plane with $x(t)$ & $y(t)$ with parametric Eqn. when this cartesian plane is visualised as cart complex plane then ϕ represents γ.

Exp. $\gamma(t) = 2t^2 + 4ti, \quad t \in [0,1]$

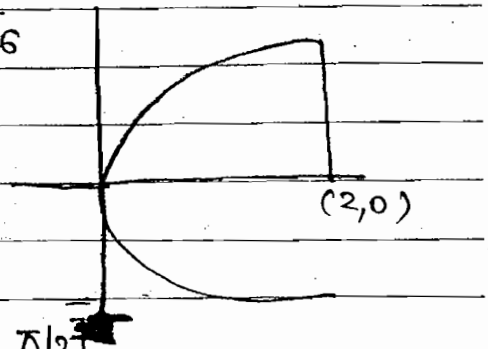
$x = 2t^2 \quad y = 4t$

$$x = 2\left(\frac{y}{4}\right)^2 = 2\cdot\frac{y^2}{16}$$

$$\boxed{y^2 = 8x}$$

$x \in [0,2]$

& $y \in [0,4]$

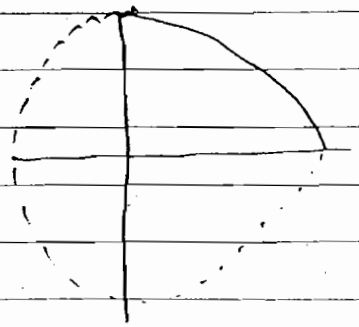


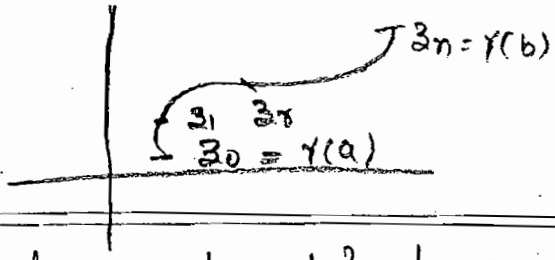
Exp. $x = a\cos t, \quad t \in [0, \pi/2]$

$y = a \sin t$

$x \in [a,0] \quad a>0$

$y \in [0,a], \quad a>0$





) If $P = \{a = t_0, t_1 \cdots t_n = b\}$ be a partition of $[a,b]$ & we denote $z_r = \gamma(t_r) = x(t_r) + iy(t_r)$ z_r represent a complex no.

;) $\gamma(a)$ & $\gamma(b)$ are called ends point of curve. they are also refered as Initial point & terminal point resp.

+) If $t_{r-1} \le t'_r \le t_r$ & denote $z'_r = \gamma(t'_r)$ then z_r' lies b/w the position of z_{-1} and z_r.

* Closed Curve :- γ is said to be closed curve if $\gamma(a) = \gamma(b)$

$\phi(x, y) = 0$

$x^2 + y^2 - 1 = 0$

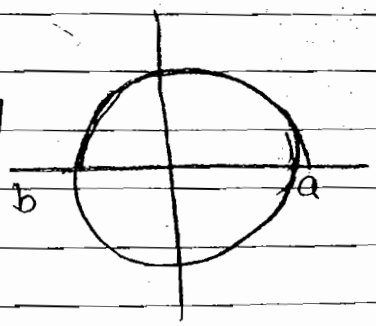
$\gamma(t) = \cos t + i \sin t$, $t \in [0, 3\pi]$

$\gamma(a) = 0$

$\gamma(b) = 3\pi$

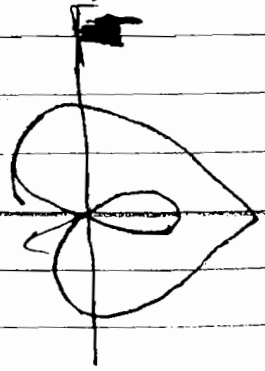
$\gamma(a) \neq \gamma(b)$

It is not closed curve.



Exp. $r = a + b\cos\theta$, $\theta \in [0, 2\pi]$, $b > a$

eg. $r = 3 + 5\cos\theta$



γ is 1-1 on open Interval.

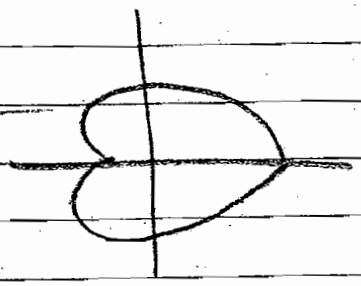
But $t_1 \neq t_2$

$\gamma(t_1) = \gamma(t_2)$

JORDAN ARC :→ If γ is 1-1 an open interval (a, b) then γ is called Jordan arc or Simple arc.

* JORDAN Curve :→ A closed Jordan arc is called Jordan curve.

Jordan curve



Smooth Curve :→ If $x(t), y(t) \in C^1[a\ b]$

$\gamma'(t) = x'(t) + y'(t) \neq 0$ for $t \in [a, b]$

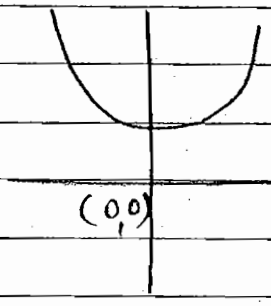
Then γ is Smooth curve.

Ex. $\gamma(t) = 4t + (4t^2+1)i \quad t \in [0,1]$

$x = 4t \qquad y = 4t^2+1$

$t = \frac{x}{4}$

$y = 4 \times \frac{x^2}{4\times 4} + 1 \quad \text{—①}$



$x^2 - 4y + 4 = 0$

$\gamma'(t) = 4 + 0ti \neq 0$

∴ it is Smooth curve.

Slope of the tangent at $t = 1/2$

$\frac{dy}{dx} = \frac{1}{2}x \qquad (\text{by } i)$

∵ $x = 4t$ & $t = 1/2$, $x = 2$

∴ $\frac{dy}{dx} = 1$

Let $\gamma : [a, b] \to \mathbb{C}$

define $\gamma : [a, b] \to \mathbb{C}$

as $\gamma_i(t) = \gamma(a+b-t)$

γ_i & γ are the same curve with reversed end points.

Ex. $\gamma(t) = e^{it} \qquad t \in [0, \pi]$

$\gamma_1(t) = \gamma(\pi + 0 - t)$

$\gamma_1(t) = e^{(\pi - t)i}$

$= e^{\pi i} \cdot e^{-ti} = -e^{-it}$

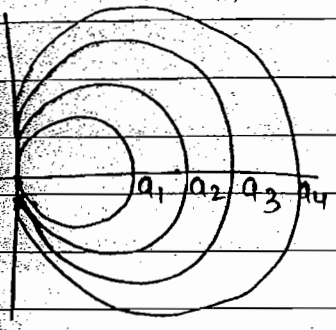
$= -\cos t + i \sin t, \quad t \in [0, 2\pi]$

Note: → γ & γ' are same but in opposite direction i.e. one is in anti clock & another is in clockwise.

WINDING No.: → clockwise → left no. outside "0" & anticlock wise → right no. outside.

Let γ be a closed curve and α be any complex no. not lying on the curve then the winding no. of γ w.r.to. α with no. of time. —
γ winds α in anticlock wise direction.

If it winds in clockwise direction then taken to be −ve.



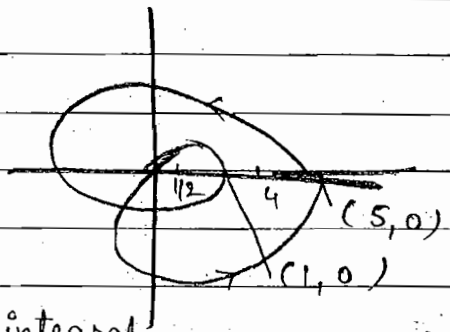
Winding no. of −
$a_1 = 4$
$a_2 = 3$
$a_3 = 4$
$a_4 = 1$
$a_5 = 0$

Exp.
$\gamma_1 = e^{it}$ $\quad t \in [0, 4\pi]$
$\gamma_2 = -e^{it}$ $\quad t \in [0, 6\pi]$.
$w(\gamma_1, 0) = 2$
$w(\gamma_2, 0) = -3$

Exp. $\gamma(t) = \cos t + i \sin t$, $t \in [0, 3\pi]$
It is not closed curve
∴ No winding No.

∴ If winding no. in anticlock wise direction = 9
clockwise = 5 (-5)
∴ Net winding no. = 9 - 5 = 4

Ex:- $\gamma = 2 + 3\cos\theta$, $\theta \in [0, 2\pi]$ =-
$\omega(\gamma_1, 1/2) = 2$
$\omega(\gamma_2, 4) = 1$ Ans.



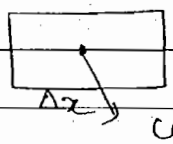
$\int_\gamma \frac{1}{(z-1/2)}\, dz = 2\pi i\, \omega(\gamma, a)$
$= 4\pi i$

$\int_a^x$ → Indefinite Integral

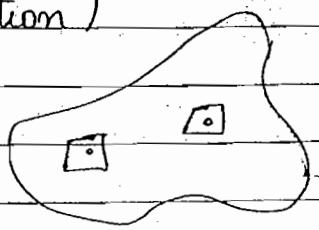
$\int_\gamma \frac{1}{(z-2)}\, dz = 2\pi i$ $\int_a^b$ → definite integral

$\int$ → primitive

Surface involution :-



cost at two pt (cost is fun^c of position)

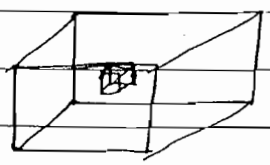


$\sum\sum f(x,y)\,\Delta x\,\Delta y \rightarrow \int_a^a \int_a^b f(x,y)\,dx\,dy$

if $f(x,y)$ then $\int_c^d \int_a^b$ → area.

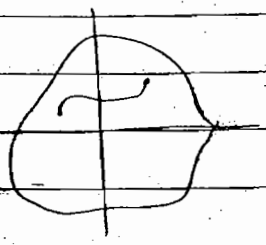
Volume involution :→

Equ. of Something along volume.



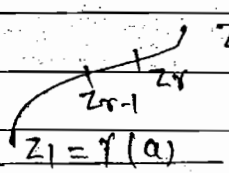
$\Delta w = f(x,y,z)\,\Delta x\,\Delta y\,\Delta z$
$\sum\sum\sum f(x,y,z)\,\Delta x\,\Delta y\,\Delta z = \iiint$

Let f is defined in D and γ be a curve in D.
defined from $[a,b] \longrightarrow \mathbb{C}$



Let P be a partition of $[a,b]$
$P = \{ t_0 = a, t_1 \cdots t_n = b \}$

$z_k = \gamma(t_k)$ is complex no. in D lying on the curve.



Define $S_n = \sum_{\gamma} f(z_r') (z_r - z_{r-1})$ $\longrightarrow$

Such type is not defined in $\mathbb{R}^2 \to$ v.s.
But here $\mathbb{C} \to \mathbb{C}$ field

where $z_r' = \gamma(t_r')$ for some $t_{r-1} \le t_r' \le t_r$
S_n is S.o.p.s. of complex no.
$\|P\| \to 0 \Rightarrow n \to \infty$
Then $\lim S_n$ if exists is defined as complex integral of f along the curve γ and denoted by

$$\int_{\gamma} f(z)\, dz$$

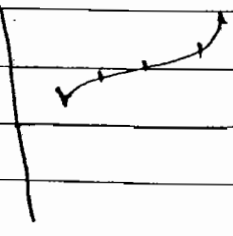
$$\left[S_n = \sum_{\gamma} f(z_r') (z_r - z_{r-1}) = \sum_{\gamma} f(z_r') \Delta z_r \right]$$

Results & properties :→

1. $dz = dx + i dy$

2. $\sum_{r=1}^{n} |\Delta z_r| \longrightarrow \int_{\gamma} |dz| \longrightarrow$ length of the curve γ.

along γ →
$z = \gamma(t)$
$dz = \gamma'(t')\, dt$



length of curve.

$$\int_\gamma |dz| = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \left(\because \gamma'(t) = \frac{dx}{dt} + i\frac{dy}{dt}\right)$$

Let $\gamma = 3\cos t + 2i\sin t$, $t \in [0, \pi/4]$

$$\int_\gamma |dz| = \int_0^{\pi/4} \sqrt{(-3\sin t)^2 + (2\cos t)^2}\, dt$$

$= \frac{\pi}{8}(3+2)$

$= \frac{5\pi}{8}$

πab = area of ellipse

$\pi(a+b)$ = perimeter

$x = 3\cos t$, $y = 2\sin t$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

1. $\int_\gamma z\, dz$

$S_n = \sum f(z_r')\,\Delta z_r = \sum f(z_r')(z_r - z_{r-1})$

Let $z_r' = z_r$, $z_r' = z_{r-1}$

$S_n = \sum z_r \Delta z_r$

$S_n = \sum z_r (z_r - z_{r-1})$

and $S_n = \sum z_{r-1} (z_r - z_{r-1})$

On adding, $2S_n = \sum z_1(z_1 - z_0) + z_2(z_2 - z_1) + \cdots + z_n(z_n - z_{n-1})$
$+ z_0(z_1 - z_0) + \cdots + z_{n-1}(z_n - z_{n-1})$

$2S_n = z_n^2 - z_0^2$

$S_n = \frac{1}{2}(z_n^2 - z_0^2) = \frac{1}{2}\left[(\gamma(b))^2 - (\gamma(a))^2\right]$

Integral is —

⇒ Independent of path

⇒ $\int_\gamma z\, dz = \frac{1}{2}(z_n^2 - z_0^2)$ $\forall\ \gamma$

⇒ $\int_\gamma f(z)\, dz = \int_\gamma f(\gamma'+1)\, dz$ ($f(z)$ and dz are governed by γ)

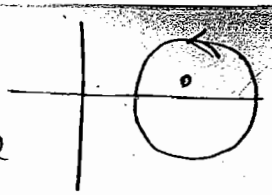
$= \int_a^b f(\gamma(t))\, \gamma'(t)\, dt$

Eg:- $\gamma : z = a + re^{it}$, $t \in [0, 2\pi]$

$f(z) = \frac{1}{(z-a)}$

$$\int_\gamma f(z)\, dz = \int_0^{2\pi} \frac{1}{re^{it}}\, rie^{it}\, dt = i\int_0^{2\pi} dt = 2\pi i$$

$\Rightarrow \int \frac{dz}{(z-a)} = 2\pi i$



Analytic, Singularity at 'a' isolated, pole

$D = \mathbb{C} - \{a\}$, $R = \mathbb{C} - \{a\}$

$R' = \mathbb{C}$, $S = \{a\} \longrightarrow$ Isolated Singularity.

$\Rightarrow$ If γ be any closed curve and 'a' lies within γ then-

$$\int_\gamma \frac{1}{(z-a)}\, dz = 2\pi i\, w(\gamma, a)$$

$$\Rightarrow w(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{1}{(z-a)}\, dz$$

* $\left| \int_\gamma f(z)\, dz \right| \leq \int_\gamma |f(z)|\, |dz|$ take care of $|dz|$.

* If $M = \sup_{z \in \gamma} |f(z)|$

Then $\left| \int_\gamma f(z)\, dz \right| \leq Ml$ where $l \rightarrow$ length

*

$f(z) = 1$, $\gamma = it$, $t \in [0,1]$

$$\int_\gamma f(z)\, dz = \int_0^1 (1)(i)\, dt = i$$

Now, $\operatorname{Re} \int_\gamma f(z)\, dz = 0$

$$\Rightarrow \int_\gamma \operatorname{Re}[f(z)]\, dz = \int_\gamma (1)\, dz = \int_0^1 i\, dt = i$$

$$\Rightarrow \operatorname{Re} \int_\gamma f(z)\, dz \neq \int_\gamma \operatorname{Re}[f(z)]\, dz$$

*

$f(z) = u(x,y) + iv(x,y)$

$dz = dx + i\,dy$

$$\int_\gamma f(z)\, dz = \int_\gamma (u+iv)(dx+i\,dy)$$

$$= \int_\gamma (u\,dx - v\,dy) + i \int_\gamma (u\,dy + v\,dx)$$

Next $\rightarrow$ Conti. in last

If $f'(x)$ is conti. on $[a, b]$
then $\int_a^b f'(x)\,dx = f(b) - f(a)$

* RESULT :—

1. If f is conti in a domain D.
$w = f(z) = u + iv$
Then the following statements are equivalent.
i) f has primitive.
ii) The integral of f along any curve joining p & q depends only at the pt. of p & q i.e. integral of f is path independent.
iii) Integral of f along any closed curve lying in D is zero.

Note :→ $a + b - t = s$
as $(-\gamma)(t) = \gamma(a+b-t)$
$\Rightarrow \int_{-\gamma} f(z)\,dz = \int_{-\gamma} f(\gamma(a+b-t))\,(\gamma'(a+b-t))\,(-dt)$
$= -\int_{\gamma} f\gamma(s)\,(\gamma'(s))\,ds$

* $F'(z) = f(z)$
$w = f(z)$
$w(t) = f[\gamma(t)] = u(t) + iv(t)$
$\int_a^b w'(t)\,dt = \int_a^b [u'(t) + iv'(t)]\,dt$
$= \int_a^b u'(t)\,dt + i\int_a^b v'(t)\,dt$
$= u(b) - u(a) + i[v(b) - v(a)]$

$$= (U(b) + iV(b)) - (U(a) + iV(a))$$

$$= F[\gamma(b)] - F[\gamma(a)]$$

$$\Rightarrow \int_\gamma f(z)\,dz = F[\gamma(b)] - F[\gamma(a)]$$

(i) $\Rightarrow$ (ii)

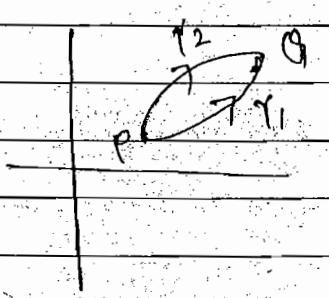
$$F'(z) = f(z)$$

$$\int_\gamma f(z)\,dz = F[\gamma(b)] - F[\gamma(a)]$$

(ii) $\Rightarrow$ (iii) $\quad \gamma(b) = \gamma(a)$

$$\Rightarrow F[\gamma(b)] - F[\gamma(a)] = 0$$

$$\Rightarrow \int_\gamma f(z)\,dz = 0$$



(iii) $\Rightarrow$ (ii)

$$C = \gamma_1 + (-\gamma_2)$$

$\Rightarrow$ C is closed curve.

given $\int_C f(z)\,dz = 0$ as C is closed

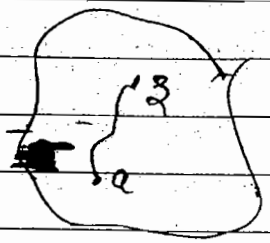
$$\int_{\gamma_1 + (-\gamma_2)} = 0 \Rightarrow \int_{\gamma_1} + \int_{-(\gamma_2)} = 0$$

$$\Rightarrow -\int_{\gamma_1} - \int_{\gamma_2} = 0$$

$$\Rightarrow \boxed{\int_{\gamma_1} = \int_{\gamma_2}}$$

(ii) $\rightarrow$ (i)

$$f(z) = \int_a^z f(s)\,ds$$



$$F(z+h) = \int_a^{z+h} f(s)\,ds$$

$$\Rightarrow \frac{f(z+h) - F(z)}{h} = \frac{1}{h}\int_z^{z+h} f(s)\,ds$$

$$\frac{f(z+h)-f(z)}{h} - f(z) = \frac{1}{h}\left[\int_z^{z+h} f(s)\,ds - f(z)\int_z^{z+h} ds\right]$$

$$\left|\frac{f(z+h)-f(z)}{h} - f(z)\right| = \left|\frac{1}{h}\left[\int_z^{z+h} f(s)-f(z)\,ds\right]\right|$$

$$\leq \frac{1}{|h|}\int_z^{z+h} |f(s)-f(z)|\,|ds|$$

$$f'(z) - f(z) \leq \varepsilon$$

$$\Rightarrow \quad f'(z) = f(z)$$

Cauchy's Thm :→ If f is analytic in D (open & connected) & $f'(z)$ is continuous in D then integral of f along any closed curve is zero.

i.e. $\oint_C f(z)\,dz = 0$

Proof:- $$\int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx+i\,dy)$$

$$= \int_\gamma (u\,dx - v\,dy) + i(u\,dy + v\,dx)$$

Green Thm :→

$$\oint M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\iint_R (-v_x - u_y)\,dx\,dx + i\iint_R (u_x - u_y)\,dx\,dy$$

$$0 + i0 = 0$$

Cauchy – Gaurset :→ If f is analytic in D then $\int_\gamma f(z)\,dz = 0$ in D.

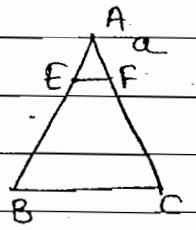
$\Rightarrow$

Corollary :→ If f is analytic in $D-\{a\}$ where $a \in D$ if f is continuous at 'a' then also integral along any closed curve is 0 in D.

$$\int_\gamma f(z)\,dz = 0$$

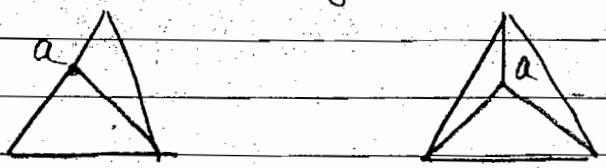
$$\left|\int_\Delta\right| = \left|\int_\square + \int_{\Delta^A_F}\right| = \left|\int_{\Delta'}\right| \le Ml < M\cdot\frac{\varepsilon}{M}$$

$$< \varepsilon$$



Vertex of mid point

Point A is at the mid point of any side of triangle = 0



* f is analytic in D
  $a \in D$
  $\Rightarrow f'(a)$ exists.

define $f : D \to C$

$$\phi(z) = \begin{cases} \dfrac{f(z)-f(a)}{z-a}, & z \neq a \\ f'(a), & z = a \end{cases}$$

$\Rightarrow$

$$\lim_{z\to a} \phi(z) = \phi(a)$$

ϕ is analytic in $D\setminus\{a\}$ & conti at $z=a$

$$\gamma = |z-a| = r$$

$$\int_\gamma f(z)\,dz = 0$$

$$\Rightarrow \int_\gamma \frac{f(z)-f(a)}{z-a}\,dz = 0$$

Cauchy Integral formula :→

$$\boxed{f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)}\,dz}$$

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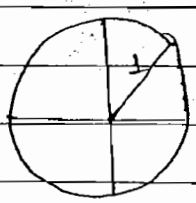
Q:- $f : D \to \mathbb{C}$

$f(z) = \text{Exp}(e^{it})$

$\int_0^{2\pi} f(z)\,dt = ?$

Sol^n :- $z = e^{it}$ $(\because r = 1)$



$f(z) = e^z$

$dz = ie^{it}\,dt$

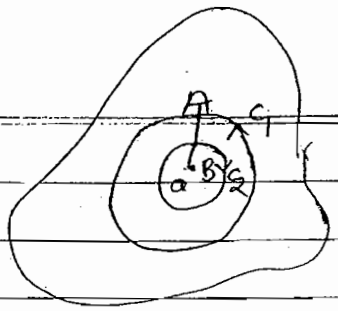
$dt = \frac{dz}{iz}$

$\frac{1}{i}\int \frac{e^z}{z}\,dz$ (By Integral formula)

$2\pi i \times \frac{1}{1} f(0)$

$= 2\pi \cdot e^0 = 2\pi$ Ans.

NOTE:→ Let f is analytic in $D-\{a\}$ where 'a' is some pt of D. and C_1 & C_2 are two curve around a. C_1 is anticlock wise & C_2 is clock wise.



Define $C = C_1 + AB + C_2 + BA$

then f is analytic with in and on C.

$$\int_C = 0$$

$$\int_{C_1+AB+C_2+BA} = 0 \Rightarrow \int_{C_1} + \int_{AB} + \int_{C_2} + \int_{BA} = 0$$

$$= \int_{C_1} = -\int_{C_2} \Rightarrow \int_{C_1} = \int_{-C_2} = \int_{C_2'}$$

Result :→ $\boxed{\int_C \frac{1}{z-a}\,dz = 2\pi i\, W(C_1, a)}$ (W = winding No.)

* $C: \frac{x^2}{9} + \frac{y^2}{4} = 1$

$$\int_C \frac{e^z}{(z-1)(z-3i)}\,dz \Rightarrow \int \frac{e^z/(z-3i)}{(z-1)}\,dz$$

$$= 2\pi i f(1) = 2\pi i \cdot \frac{e}{(1-3i)}$$

$2 \le |z| < 3$

$\therefore$ $3i$ does not lies in circle

$\therefore$ we write $\frac{e^z/(z-3i)}{(z-1)}$

if 1 & $3i$ both lies inside circle

$\therefore$ $\frac{x^2}{9} + \frac{y^2}{25} = 1$

Then $\frac{e^z}{(z-1)(z-3i)} = \frac{e^z}{(1-3i)}\left[\frac{1}{(z-1)} - \frac{1}{(z-3i)}\right]$

NOTE! -

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)}\,ds$$

$$\boxed{f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\,ds}$$

By Induction

$$\boxed{f^n(z) = \frac{1^n}{2\pi i}\int_C \frac{f(s)}{(s-z)^{n+1}}\,ds}$$

Q. $\int_C \frac{\cos z}{z^2}\,dz$

$\Rightarrow 2\pi i f'(0) = 0$

Q:- $\int \frac{\cos z}{z^{2014}} = 0$

Q. $\int \frac{\sin z}{z^{10}} = \frac{2\pi i f^9(0)}{9!} = \frac{2\pi i}{9!}$

$n = \frac{9-1}{2} = 8 = +1$

$f(0) = 0,\ f'(0) = 1$

$f'''(0) = -1,\ f^5(0) = 1$

NOTE:→

1) If f is analytic then all the derivatives of f are analytic.

2) If f has primitive then iff it is analytic.

*

Morera's Thm :→ "If f is continuous in D and Integral of f along any closed curve is zero. i.e. f is analytic in D."

this thm is converse of Cauchy Thm.

* At any $a \in D$

we have, $f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$

where C is a closed curve in D & f is analytic in D.

Now if $C: |z-a| = R$ Simple closed curve

$$|f^n(a)| \le \frac{n!}{2\pi} \int_C \frac{|f(z)|}{R^{n+1}} |dz|$$

if $M = \text{Sup } |f(z)|$, $z \in C$

$$|f^n(a)| \le \frac{n!\,M}{2\pi R^{n+1}} \int_C |dz| \qquad (\because |dz| = 2\pi R)$$

$$\Rightarrow \boxed{|f^n(a)| \le \frac{n!\,M}{R^n}} \rightarrow \text{Cauchy Inequality.}$$

Exp. $f(z) = \frac{(z^2+3)\, e^{iz} \log z}{z^2-2}$

$C: |z| = 2 \quad C = \{ z : z = 2e^{i\theta} \quad \theta \in [0, \pi/3] \}$

$$|f(z)| = \frac{|z^2+3|\,|e^{iz}|\,|\log z|}{|z^2-2|} \qquad (\because z = re^{i\theta},\ r = 2)$$

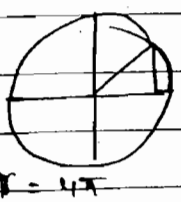
$$\le \frac{(|z|^2+3)\, e^{y}\, (\log 2 + \theta)}{|z|^2-2|}$$

$$M = \frac{7(3\log 2 + \pi)}{6} \qquad (\because \theta = \pi/3)$$

if $\theta \in [0, 2\pi]$ then min. value of $y = -1$

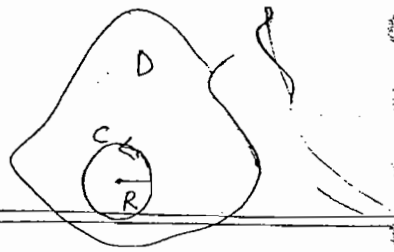
$\therefore e^y = e^{(-1)} = e$ //

$$\left| \int_C f(z)\, dz \right| \le \frac{7(3\log 2 + \pi)}{6} \cdot \frac{2\pi}{3}$$



$2\pi r = 4\pi$ $\int_0^{\pi}$

$\int_0^{\pi/3} \frac{2\pi}{3}$ //



$f: D \to C$

f is analytic

$|f^n(a)| \leq \frac{n!}{R^n} M$

if $|f(z)| \leq 1 \quad \forall \; z \in D$

$|f^n(1)| \leq \frac{1 \cdot L^n}{R^n}$

$$\boxed{|f'(a)| \leq \frac{1}{R}}$$

Entire function :→ $f: C \to C$ is said to be entire if it is analytic on C.

Note:- ∞ is only possible singularity of entire function.

Some common entire fun^c :→

1) $f(z) = p(z) = \sum a_i z^i$
2) $f(z) = \sin z$
3) $f(z) = \cos z$
4) $f(z) = e^z$
5) $f(z) = e^{g(z)}$, $g(z)$ is entire
6) $f = \alpha g + \beta h \quad \forall$ entire $g \& h$
7) $f = g \cdot h$

Note:- 1) The set of all entire fun^c w.r.to point wise addition & multiplication form ring & this ring CRU without zero division.

2) $f(z) = 1$ is which is unity
$e(z) = 1$, units of this ring are only those which

entire fun^c which has no zero in C.

* No polynomial can be units.
* linear combination of entire fun^c is Entire.
* It is not U.F.D. so not E.D., P.I.D.
  ↓ This ring
* Non-constant linear polynomials are irre element.
* The set of all Entire fun^c forms V.S. over the field of $\mathbb{C}$ under two composition —
  $(f+g)(z) = f(z) + g(z)$
  $(\alpha \cdot f)(z) = \alpha \cdot f(z) \quad \forall \alpha \in \mathbb{C}$
  it is not FDVS.

Let f is entire.
  $D \subset C$, D is bounded
  then f is bounded in D.

Liouville's THm :→ "Every Entire and bounded fun^c is Constant."

* If f is non-constant Entire fun^c ⇒ f is unbdd.

Q. $f : \mathbb{C} \xrightarrow{\text{analytic}} \mathbb{C}$
$f(0) = 0, \quad f(1) = 1$
f is entire & non-constant (i.e. unbdd)

* $\sin z$ is unbdd fun^c.

C_6:- If f is analytic in $\mathbb{C}$ & bdd in a nbd of ∞ then f is constant.

Proof:- $|f(z)| \leq \lambda \quad \forall \; z \in \mathbb{C}$

$$|f'(a)| \leq \frac{M}{R} =$$

$$\leq \frac{\lambda}{R} < \varepsilon$$

$$f'(a) = 0 \Rightarrow f'(z) = 0 \quad \forall \; z \in \mathbb{C}$$

$$\boxed{f(z) = c \quad \forall \; z \in \mathbb{C}}$$

Application of Liouville's Thm:→

1) If any of Real & Img. part of the entire fun^c is bdd then f is const.

Exp.

$$f(z) = u + iv$$

$$|g(z)| = |e^{f(z)}| = e^u < e^M$$

$$|h(z)| = |e^{if(z)}| = e^v < e^M$$

Q:→ If $au + bv \leq c$ then also f is constant.

Exp.

$$g(z) = e^{(a-ib)(u+iv)}$$

$$|g(z)| = e^{(au+bv)} < e^c$$

C_9:- In any of the following condition fun^c is constant —

(a) $u(x,y) \geq c$

(b) $v(x,y) \geq c$

(c) $au + bv \geq c$

$$|g(z)| = |e^{-f(z)}| = e^{-u} = \frac{1}{e^u} < \frac{1}{e^c}$$

$$|h(z)| = |e^{if(z)}| = e^{-v} = \frac{1}{e^v} < \frac{1}{e^c}$$

$$|f(z)| = |e^{-(a+ib)f(z)}| = e^{-(au+bv)}$$

$$= \frac{1}{e^{(au+bv)}} < \frac{1}{e^c}$$

* If $w = f(z) = u+iv$ is an entire fun^c and there is a line in w-plane s.t. at any pt z, $f(z)$ lies always above that line (below that line), then fun^c is constant.

C_{10} :→ If f is entire s.t.
$|f(z)-\alpha| > b \neq 0$

define $h(z) = \frac{1}{f(z)-\alpha}$ (f is entire $\therefore h$ is entire)

$$|h(z)| = \left|\frac{1}{f(z)-\alpha}\right| < \frac{1}{b}$$

h is bounded.
$\therefore h$ is constant.

Q:- If f & g are entire fun^c s.t. $|f| < |g|$ $\forall z \in C$
Then f & g are L.D. on C.

Soln :- $|h(z)| = \left|\frac{f}{g}\right| < 1$

h is entire & bdd so constant
$\therefore \left|\frac{f}{g}\right| < 1 \Rightarrow |f(z)| = d\,|g(z)|$
L.D.

=11:- If f is entire and has two L.I. periods over $\mathbb{R}$ then f is constant

i.e. if $z_1, z_2 \in \mathbb{C}$ are L.I. over $\mathbb{R}$

i.e. $\frac{z_1}{z_2} \notin \mathbb{R}$

and $f(z+z_1) = f(z+z_2) = f(z)$

$\forall\ z \in \mathbb{C}$

Then f is constant.

Q: $f(z+1) = f(z+i) = f(z) \quad \forall\ z \in \mathbb{C}$

$\Rightarrow f$ is constant.

22 | Nov | 2014

Fundamental THm of algebra in Complex :→

" Every non-constant polynomial over $\mathbb{C}$ has exactly n zeros in $\mathbb{C}$ counting multiplicity where n is degree of polynomial. $(n \in \mathbb{N})$"

$P(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad a_n \neq 0$

Let $f(z)$ has no zero in $\mathbb{C}$.

$f(z) = \frac{1}{P(z)}$ is Entire.

$f(z) \Rightarrow \frac{1}{a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0}$

$f(z) = \frac{1}{a_n R^n \cdot e^{ni\theta} + a_{n-1} R^{n-1} e^{(n-1)i\theta} + \cdots a_0}$

$= \frac{1}{\left| R^n e^{ni\theta} \left[a_n + \frac{a_{n-1}}{R} e^{-i\theta} \cdots \frac{a_0}{R^n} e^{-ni\theta} \right] \right|}$

$|f(z)| = R^n |\ldots| \quad$ for large R

$|f(z)| < 1$

$\Rightarrow$ f is constant

$\Rightarrow$ $p(z)$ is constant $\quad \times$ ⟹

our assumption is wrong.

i.e. $p(z)$ has atleast one zero in $\mathbb{C}$.

$p(z) = (z-\alpha_1)\, p_1(z)$

$$\boxed{p(z) = a_n \prod (z-\alpha_i)}$$

II nd thm :→

Gauss - THEOREM

Proof :→

Let $p'(\alpha) = 0$ & $p(\alpha) \neq 0$

where $p(z) = a_n \prod_{i=1}^{k} (z-\alpha_i)$

Taking log and differentiate.

$$\frac{p'(z)}{p(z)} = \sum \frac{1}{(z-\alpha_i)}$$

$$\Rightarrow \sum \frac{1}{\alpha-\alpha_i} = 0 \qquad \sum \frac{1}{\alpha-\alpha_i} = 0$$

& $p'(\alpha) = 0$ & $p(\alpha) \neq 0$

$$\sum \frac{\bar{\alpha} - \bar{\alpha_i}}{(\alpha-\alpha_i)(\bar{\alpha}-\bar{\alpha_i})} = 0$$

$$\Rightarrow \sum \frac{\bar{\alpha} - \bar{\alpha_i}}{|\alpha-\alpha_i|^2} = 0$$

$$\Rightarrow \sum \frac{\bar{\alpha}}{|\alpha-\alpha_i|^2} = \sum \frac{\bar{\alpha_i}}{|\alpha-\alpha_i|^2}$$

one of the coff. of P(z) should be complex.

a+ib
a-ib

$$\Rightarrow \sum \frac{\alpha}{|\alpha-\alpha_i|^2} = \sum \frac{\alpha_i}{|\alpha-\alpha_i|^2}$$

Let $\lambda = \sum \frac{1}{|\alpha-\alpha_i|^2} > 0$

$$\Rightarrow \alpha \cdot \lambda = \sum \frac{\alpha_i}{|\alpha-\alpha_i|^2}$$

$$\alpha = \sum \frac{\alpha_i}{\lambda\,|\alpha-\alpha_i|^2}$$

$$= c_1\alpha_1 + c_2\alpha_2 + \cdots c_k\alpha_k$$

$$c_i = \frac{1}{\lambda\,|\alpha-\alpha_i|^2} > 0$$

$$\sum c_i = \frac{1}{\lambda} \sum \frac{1}{|\alpha-\alpha_i|^2} = 1 \qquad \left(\because \frac{1}{|\alpha-\alpha_i|^2} = \lambda\right)$$

Hence $\alpha = \sum c_i\alpha_i$

where $c_i > 0$ & $\sum c_i = 1$

"Zero's of $P'(z)$ are convex combination of zero's of $P(z)$"

Statement of Cauchy Thm.

Zero's of $P'(z)$ lies on the convex hull of the set of $P(z)$.

"The smallest convex set containing S is called Convex hull.
And denoted by $H(S)$."

*

Luca's Thm :→ If Zero's of $P(z)$ lies in the upper half of complex plane then that of $P'(z)$ also lie on same half plane.

Unit - 4

Power Series :→ $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, $a_n \in \mathbb{C}$

is defined as formal power series | power series at z_0.

$\Rightarrow a_0 + a_1(z-z_0) + a_2(z-z_0)^2$

or $z_0 = 0$

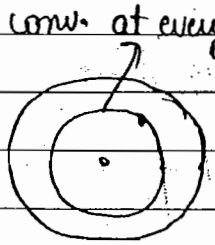
we have $\sum_{n=0}^{\infty} a_n z^n$

Results & properties :→

1) Power Series is a series of functions & each fun^c of this series is a polynomial.

2) for any $z = \alpha \in \mathbb{C}$
we get a series of complex no.'s namely,
This series at $z = z_0$ is always convergent and converges to a_0.

3) If $\exists\ \alpha \in \mathbb{C}$ s.t. $\alpha \neq z_0$
s.t. $\sum a_n (z-z_0)^n$ is convergent
then the series is convergent.
$\forall\ z \in |z-z_0| < |\alpha - z_0|$



4) The largest R s.t. series convergent at every pt of $|z-z_0| < R$
$\Rightarrow$ & defined as the radius of convergence of disc.

5) The convergence is uniform on $|z-z_0| \leq r < R$
and absolute $|z-z_0| < r < R$.

If $\sum a_n (z-z_0)^n = f(z)$
& $\forall z \in |z-z_0| < R$
& f is the uniform limit of $\sum a_n (z-z_0)^n$ in $|z-z_0| \le r < R$.

Now, Let $D = |z-z_0| < R$
& C is any closed curve in D.

$$\int_C f(z)\,dz = \int_C \sum a_n (z-z_0)^n \,dz$$

$$= \sum a_n \int_C (z-z_0)^n \,dz$$

$$= 0$$

moreover as, $a_n (z-z_0)^n$ is conti. $\forall n \in \mathbb{N} \cup \{0\}$
and convergence is uniform
$\Rightarrow f$ is continuous.
$\Rightarrow f$ is Analytic. (By morera's thm)

$$\boxed{f(z_0) = a_0}$$

24/Nov/14.

Taylor's Series :→ Let f is analytic in D & $z_0 \in D$

then $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ ——(1)', $z \in |z-z_0| < \delta$ in D

where $a_n = \frac{f^n(z_0)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$

$C : |z-z_0| = r < \delta$

Properties & Results :→

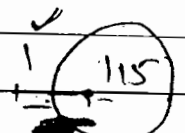
1) The Taylor series of f in a nbd of z_0. & Conversly, f is considered as unifor the limit of the power series of eau (1).
The Sum or limit of the converge is unique.

2) Taylor series converges to uniformly in $|z-z_0| < R$ where R is the near distance of nearest sing. of f from the point of expansion z_0.
At every pt of this series is divergent in $|z-z_0| > R$ & $|z-z_0| = R$, nothing can be said precessiy.

Exp. $f(z) = \frac{1}{(2z-1)(z-1)}$

at $3/2$

Roc $= 1.5 - 1 = 1/2$ ⟹



$\star$ $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, $|z-z_0| < R$

1) uniformly conti. on $|z-z_0| \leq r < R$
2) Absolute conv. $|z-z_0| < r < R$
3) divergent on $|z-z_0| > R$

-) Can't say on $|z-z_0| = R$.

Properties :- (*)

Zero's of Non-constant analytic fun^c :-

Let f is analytic on D and if f is Non-constant $z_0 \in D$ is said to be zero of order m if -

$$f(z_0) = f'(z_0) = f''(z_0) \cdots = f^{m-1}(z_0) = 0$$
$$\& \quad f^m(z_0) \neq 0$$

z_0 is called zero of order m.

Zero's of Non-constant analytic fun^c f is isolated.

-) Let z_0 is a zero's of order m then

$$f(z_0) = f(z_0) \cdots f^{m-1}(z_0) = 0$$
$$f^m(z_0) \neq 0$$

& if $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

$$\Rightarrow a_0 = f(z_0) = 0$$
$$a_1 = \frac{f'(z_0)}{1!} = 0$$
$$a_r = \frac{f^r(z_0)}{r!} = 0, \quad r = 1 \text{ to } m-1$$
$$a_m = \frac{f^m(z_0)}{m!} \neq 0$$

$$f(z) = (z-z_0)^m \left[a_m + a_{m+1}(z-z_0) + \cdots \right]$$
$$= (z-z_0)^m \sum_{n=0}^{\infty} c_n (z-z_0)^n, \quad c_n = a_{m+n}$$

where $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ is a power series and convg. on $|z-z_0| < R$

Let $\sum c_n (z-z_0)^n = g(z)$

then $g(z)$ is analytic in $|z-z_0| < R$

& $g(z_0) = c_0 = a_m \neq n$

$\Rightarrow$ $\exists\ \delta > 0$ s.t. $g(z) \neq 0$

$\forall\ z \in |z-z_0| < \delta$

$z_n \in |z-z_0| < 1/n$

$\Rightarrow \langle g(z_n) \rangle \to 0 = g(z_0) \neq 0$ ✗

when $\langle z_n \rangle \to z_0$

$\exists\ \delta > 0$ s.t. $g(z) \neq 0\ \forall\ z \in |z-z_0| < \delta$

i.e. in $|z-z_0| < \delta$, $\frac{1}{g(z)}$ is analytic

$\Rightarrow$ in $|z-z_0| < \delta$

$f(z) = (z-z_0)^m \cdot g(z)$ has only one zero.

i.e. z_0

Conclusion,

If z_0 is a zero of order m of f then $\exists$ a non vanishing & analytic fun^c g in a nbd of z_0 s.t. $f(z) = (z-z_0)^m \cdot g(z)$

$z \in |z-z_0| < \delta$

3) If f & g are analytic fun^c having common zero at z_0 of order m & n respectively then $f \cdot g$ will have zero of order $m+n$.

Exp. $f(z) = z^2$

$g(z) = \sin^2 z$

Both have '0' common zero.

$f'(z) = 2z$, $f''(z) \neq 0$

$g'(z) = 0$, $g''(z) \neq 0$

$f(z) \cdot g(z) = 2+2 = 4$.

Q. $f(z) = \frac{1}{z - \sin z}$

$g(z) = z - \sin z$

$g(0) = 0, \quad g'(0) = 0, \quad g''(0) = 0$

$g'''(0) \neq 0$

$g(z)$ has order of 3.

$f(z) = \frac{1}{z^3 \cdot \phi(z)}$

It is pole sing. of order 3.

Rule :→ Let f & g are analytic function.

& $h(z) = \frac{f(z)}{g(z)}$

Then h is analytic except at $g(z) = 0$

Now let $g(\alpha) = 0$ of order m.

then α is a singularity of h.

(i) $f(\alpha) \neq 0 \Rightarrow \alpha$ is pole of order m.

ii) if $f(\alpha) = 0$ of order k then —

a) $k \geq m$ then removable sing.

b) $k < m$ then pole of order $m-k$.

For a non-constant analytic fun^c.

* Set of zero's is closed set.

* Set of non-zero's is open set.

Q. $T = \{ z \in D : f(z) \neq 0 \}$

$T \subset D$

if $z_0 \in T \Rightarrow f(z_0) \neq 0$

$\exists \ \delta > 0$ s.t. $f(z) \neq 0 \quad \forall \ z \in |z - z_0| < \delta$

$\Rightarrow |z - z_0| < \delta \subset T$

$\Rightarrow T$ is open.

V. Imp.

IDENTITY THM :→

Let $f : D \to \mathbb{C}$

f is analytic in D.

Let $N = \{ z \in D : f(z) = 0 \} \Rightarrow N \subseteq D$

Let $\alpha \in N'$ in $\mathbb{C}$

Case-1 :- Let $\alpha \in D$ i.e. f is regular at $z = \alpha$

as $\alpha \in N'$

$\Rightarrow \exists \ \langle z_n \rangle \to \alpha, \quad z_n \in N$

$\langle f(z_n) \rangle \to 0 = f(\alpha)$ ✗.

$f(z) = \sum a_n (z-\alpha)^n, \quad |z-\alpha| < \delta$

Let $z_n \in N \Rightarrow f(z_n) = 0$

s.t. $\langle z_n \rangle \to \alpha$

$\lim_{n\to\infty} f(z_n) = \lim \sum a_n (z_n - \alpha)^n$

$0 = a_0 = f(\alpha)$

$\Rightarrow \dfrac{f(z) - f(\alpha)}{z - \alpha} = a_1 + a_2 (z-\alpha) + \cdots$

along $\langle z_n \rangle$; $0 = a_1$

So, $a_1 = f'(\alpha) = 0$

Hence, by induction

$\Rightarrow a_n = 0 \quad \forall \ n \in \mathbb{N} \cup \{0\}$

$\Rightarrow f(z) \equiv 0 \quad \forall \ z \in |z-\alpha| < \delta$

$\Rightarrow$ $\boxed{f(z) \equiv 0 \quad \forall \ z \in D}$

Statement :→ If f is analytic in D & $f \not\equiv 0$ $\forall \ z \in N \subseteq D$. & N has limit pt in D then is identity zero in D.

& hence f is constant.

Corollary: → If f & g agree on subset of D (D is Domain of analyticity of f & g) if that subset has limit pt in D then

$$\boxed{f = g} \quad \forall\ z \in D$$

i.e. f & g agree on D.

Exp. $h(z) = f(z) - g(z)$, $z \in D$

$N \subseteq D$

$f(z) = g(z) \quad \forall\ z \in N$

$\therefore\ h(z) = 0 \quad \forall\ z \in N$

#. f & g are entire

S.t. $f(1/n) = g(1/n) \quad \forall\ n \in \mathbb{N}$

$\Rightarrow f(z) = g(z) \quad \forall\ z \in \mathbb{C}$

lim pt of $\frac{1}{n} = 0$, $0 \in \mathbb{C}$

Exp.

$f(z) = \sin^2 z + \cos^2 z$

$g(z) = 1$

$f(z) = g(z) \quad \forall \quad z \in \mathbb{R}$

$\sin^2 z + \cos^2 z = 1$, $z \in \mathbb{R}, \mathbb{C}$

when z is real then $= 0$

$N = \mathbb{R}$

Q. $A = \{ f : f$ is entire$\}$, which vanish on $S = \{ 1/n : n \in \mathbb{N} \}$ only

Then $|A|$ is .

a. 0 — if only not given then $|A| = 1$

b. 1

c. $\aleph_0$

d. C

\# f is analytic in D

$f(\phi(n)) = \psi(n)$, $n \in \mathbb{N}$

Take $z = \phi(n)$

& $n = \phi_1(z)$

define $g(z) = \psi(\phi_1(z))$

find domain of analyticity of g.

Ex. (i) $f(1/n) = \frac{1}{n+2}$

(ii) $f(1/n) = \frac{1}{2n+1}$

(i) $z = 1/n \Rightarrow n = 1/z$

$n = \phi_1(z) = 1/z$

$g(z) = \psi\left(\frac{1}{z}\right)$

$= \frac{1}{1/z+2} \Rightarrow \frac{z}{2z+1}$

$2z+1 = 0 \Rightarrow z = -1/2$

$|z| < 1/2$

So, no f such func which is analytic in $|z|<1$

Ans → 0.

(ii) $g(z) = \frac{1}{2\times\frac{1}{z}+1} = \frac{z}{2+z}$

$f(1/n) = g(1/n)$, $z = -2$

$|z| < 2$

So func is analytic at $|z|<1$ // Ans (1).

$f(1/n) = f(-1/n) = 1/n^2 \longrightarrow f(z) = z^2$

$f(1/n) = f(-1/n) = 1/n^3 \longrightarrow$ No such f exist

$f\left(1 + \frac{\sqrt{2}}{n} i\right) = \frac{1}{n^2}$

$$\frac{\sqrt{2}}{n} i^0 = z - 1$$

$$n = \frac{\sqrt{2} i^0}{z-1}$$

$$\boxed{g(z) = -\frac{1}{2}(z-1)^2} \longrightarrow \text{Analytic} \longrightarrow \text{Entire}$$

* Let f is analytic in $D - \{a\}$ where $a \in D$

& $|f(z)| \leq M \quad \forall \; z \in 0 < |z-a| < \delta$

Define $g(z) = \begin{cases} (z-a)^2 f(z), & z \neq a \\ 0, & z = a \end{cases}$

$\Rightarrow$ g is analytic in $D - \{a\}$

$$\lim_{z \to a} \frac{g(z) - g(a)}{z-a} = \lim_{z \to a} \frac{(z-a)^2 f(z)}{(z-a)} = 0$$

$$= g'(a)$$

$\Rightarrow$ g is analytic in D.

$$g(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \quad |z - a| < \delta$$

$$a_n = \frac{g^n(a)}{n!}$$

$\Rightarrow$ $a_n = 0$ & $a_1 = 0$

$$g(z) = (z-a)^2 \sum_{n=0}^{\infty} c_n (z-a)^n, \qquad c_n = a_{n+2}$$

$$(z-a)^2 \cdot f(z) = (z-a)^2 \sum_{n=0}^{\infty} c_n (z-a)^n$$

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \qquad z \neq a$$

Let $\sum c_n (z-a)^n = \phi(z)$

ϕ is analytic in $|z-a| < \delta$

$\phi(a) = c_0$

Hence, $\boxed{\lim_{z \to a} f(z) = c_a}$

At a, f has removable sing, or f is regular at 'a'

This is called "Riemann's Thm."

In a nbd of singularity

Conclusion →

If $z = z_0$ is an isolated sing. of f then it is removable sing. iff f is bdd in a nbd of z_0"

⇒ Converse of above statement is Riemann Thm.

CASORATI - WEIERSTRASS :→

* Let z_0 be an isolated sing essential singularity of f then for any $\alpha \in \mathbb{C}$

$\exists$ $\langle z_n \rangle$ s.t. $\lim z_n = z_0$

& $\lim f(z_n) = \alpha$

i.e. $|z_n - z_0| < \delta$

$\Rightarrow |f(z_n) - \alpha| < \delta$

for that particular seq. $\langle z_n \rangle$

$A = \{ \langle z_n \rangle : \lim z_n = z_0 \}$

$B_{z_0} = \{ \lim f(z_n) \text{ if exists} : \langle z_n \rangle \in A \}$

then $B_{z_0} = C_\infty$ if z_0 is an isolated essential sing.

Exp. $f(z) = e^{1/z} \quad z \neq 0$

at $z = 0$, $f(z)$ has isolated essential sing.

Casorati - Weierstrass : →

Let z_0 be an isolated essential sing. of f then for any $\alpha \in \mathbb{C}$

$\exists \ \varepsilon > 0$ s.t.

$|f(z) - \alpha| > \varepsilon \quad \forall \ z \in |z - z_0| < \delta$

define $g(z) = \dfrac{1}{f(z) - \alpha}$

$|g(z)| < \dfrac{1}{\varepsilon} \quad \forall \ z \in |z - z_0| < \delta$

z_0 is either regular point or R.S. of $g(z)$ (Riemann)

$f(z) = \dfrac{1}{g(z)} + \alpha$

$\lim\limits_{z \to z_0} f(z) = \lim\limits_{z \to z_0} \dfrac{1}{g(z)} + \alpha$

$\Rightarrow$ z_0 is either R.S. or pole.

Laurent Series :→

Let z_0 be an isolated singularity of f then f can be express as Sum of the limits of two series if as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=0}^{\infty} b_n (z-z_0)^{-n} \quad \text{---(1)}$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{---(2)}$$ when $a_{-n} = b_n \; \forall n \in \mathbb{N}$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz$$

1. $r < |z-z_0| < R$

This Series is valid in this annulus region. where r, R are obtain in a specific manner as Domain of convergence of two Series.

2. The closed curve of integral $C: |z-z_0| = t$

$r < t < R$

3. The Expansion has 2 Series.

a) in +ve power of $(z-z_0)$ called Regular part of Expansion.

b) -ve power of $(z-z_0)$ called principal part.

The Series (1) & (2) converges to f uniformly in the domain of Converges.

4. $$\frac{1}{2\pi i} \int_C f(z) \, dz = \int \sum + \int \sum$$

$$= \sum \int_C + \sum \int_C$$ (because of U.C.)

$$\int_C (z-z_0)^n dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = 1 \end{cases}$$

5) $$\frac{1}{2\pi i}\int_C f(z)\,dz = \frac{1}{2\pi i}\sum a_n \int_C (z-z_0)^n dz + \frac{1}{2\pi i}\sum b_n \int_C (z-z_0)^{-n} dz$$

$$= 0 + b_1 = b_1$$

Called the residue of f at $z = z_0$.

$$\boxed{b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz}$$

$$\int_C f(z)\,dz = 2\pi i \cdot \text{Res}\,(f(z);\ z = z_0)$$

where C is any simple closed curve around z_0.

Singularity :— 1) If principal part is absent then Removable sing.

2) Pole : $\exists\ m \in \mathbb{N}$
s.t. $b_n = 0\ \forall\ n > m$
& $b_m \neq 0$
m is order of pole.

3) Essential : , $b_n \neq 0$ for infinite values of n.

If z_0 is a R.S. of f i.e.
$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \cdot\ z \neq z_0 \text{ i.e. } 0 < |z-z_0| < \delta$$

if $\sum a_n (z-z_0)^n = g(z)$, $|z-z_0| < \delta$

$g(z_0) = a_0$

g is analytic in $|z - z_0| < \delta$

$|a| < |b|$ $\quad \frac{1}{a-b} = \frac{1}{-b(1-a/b)}$

$= -\frac{1}{b}\left(1 - \frac{a}{b}\right)^{-1}$

$= -\frac{1}{b}\left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \cdots\right]$

$|a| > |b|$, $\frac{1}{a-b} = \frac{1}{a}\left[1 + \left(\frac{b}{a}\right) + \left(\frac{b}{a}\right)^2 + \cdots\right]$

\# $\boxed{\int_C e^{1/z^3}\, dz = 0}$

\# $\int_{|z|=1} f(z)\, dz = 0$

if $f(-z) = f(z)$

\# $\boxed{\int \frac{\cos z}{z^{2014}}\, dz = 0}$

\# $\boxed{\int \frac{\sin z}{z^{2013}}\, dz = 0}$

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$z_0 \in S - S'$, $z \in N'$, $N = \{z \in \mathbb{C} : f(z) = 0\}$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$\Rightarrow$ z_0 is R.S.

$\sum a_n (z - z_0)^n = g(z)$

$f(z) = g(z) \quad \forall \; z \in |z - z_0| < \delta - \{z_0\}$

$g(z) = 0 \quad \forall \; z \in N$

g is analytic at z_0

$\Rightarrow g(z) = 0 \quad \forall \quad z \in |z - z_0| < \delta$

$\Rightarrow f(z) = 0 \quad \forall \quad z \in |z - z_0| < \delta$ ✗

Exp. $$\frac{\cos z}{z^2} = \frac{1}{z^2}\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right]$$

$$= \frac{1}{z^2} - \frac{1}{12} + \frac{z^2}{14} - \cdots$$

Finite term

$\therefore$ Pole Sing.

Exp. $$f(z) = |z|^2 + \frac{1}{\bar{z}}$$

$$z\bar{z} + \frac{\bar{z}}{|z|^2}$$

$D = \mathbb{C} - \{0\}$

$\frac{\partial f}{\partial \bar{z}} = z + \frac{1}{|z|^2} \neq 0$, Not diff. $\Rightarrow$

$R = \phi$

$R' = \phi$

$\therefore S = \phi$

Exp. $f(z_n) = z_0 = re^{i\alpha}$

$z_n = (2n\pi + \alpha)i + \log r$

$e^{z_n} = re^{\alpha i} = z_0$

NOTE:→ Every unbdd. sequence has a divergent sub seq.

Exp. $(-1)^n \cdot n$ has no divergent sub seq.

Let $z_0 \in S - S'$ is a pole of order m.

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}, \quad b_m \neq 0$$

$$r < |z-z_0| < R$$

Now,

$$(z-z_0)^m f(z) = \sum a_n (z-z_0)^{n+m} + b_m + \cdots b_{m-1}(z-z_0) + \cdots + b_1 (z-z_0)^{m-1}$$

$$= \sum_{n=0}^{\infty} d_n (z-z_0)^n$$

where $d_n = \begin{cases} b_{m-n}, & n < m \\ a_{n-m}, & n \geq m \end{cases}$

$= \phi(z)$ analytic in $|z-z_0| < \delta$

& $\phi(z_0) - d_0 = b_m \neq 0$

$f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$, where ϕ is analytic & non-vanishing in $|z-z_0| < \delta$ for some $\delta > 0$

$\phi(z) = \sum c_n (z-z_0)^n$

as ϕ is analytic

$$f(z) = \frac{1}{(z-z_0)^m} \sum c_n (z-z_0)^n$$

$$f(z) = \underbrace{\frac{c_0}{(z-z_0)^m} + \frac{c_1}{(z-z_0)^{m-1}} + \cdots + \frac{c_{m-1}}{(z-z_0)}}_{\text{finite } -ve (z-z_0)} + c_m + c_{m+1}(z-z_0)$$

$\Rightarrow z_0$ is pole of order m. as $c_0 \neq 0$.

$$\text{Res}[f(z), z=z_0] = c_{m-1}$$

$$= \frac{\phi^{m-1}(z_0)}{\lfloor m-1}$$

$$\lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)$$

This formula used for find residues.

\# $\dfrac{1}{z - \sin z} = \dfrac{1}{z^3 \cdot \phi(z)}$

$$= \frac{1/\phi(z)}{z^3} = \frac{\psi(z)}{z^3}$$

$$\text{Res}[f(z), z=0] = \frac{1}{2!} \frac{d^2}{dz^2} \left(\frac{z^3}{z - \sin z} \right)$$

* $\dfrac{f(z)}{g(z)}$

Case-1: Test wheather f & g are entire.

Step-I: $g(z) = 0$, $z = \alpha_i \cdots$ of order m_i.

Step II :- $f(\alpha_i) = 0$ or not

(a) $f(\alpha_i) \neq 0$

Pole of order same as that of zero.

(b) $f(\alpha_i) = 0$ of order n_i

R.S. $n_i \geq m_i$

Pole $n_i < m_i$ of order $m_i - n_i$

Exp. If $f(z) = \frac{1}{z-1} + \frac{3}{z-2} + \frac{5}{2z-1}$

$= \frac{1}{t} + \frac{3}{t-1} + \frac{5}{2t+1}$

⊗ Meromorphic function :→

Let f is entire $\exists\ R$ s.t.

$\boxed{|f(z)| \leq K|z|^{\alpha}}$, $\alpha \geq 0$

$|z| > R$

if $\alpha = 0$ then f is constant.

then $f(z) = a_0 + a_1 z + \cdots + a_{[\alpha]} z^{[\alpha]}$

if $\alpha = \infty$

then $f(z)$ has pole sing.

f has atmost $[\alpha]$ degree.

If $\alpha = 2$, then f has atmost 2 degree either 0 or 1.

* f is entire.

$\Rightarrow f(z) = \sum a_n z^n$

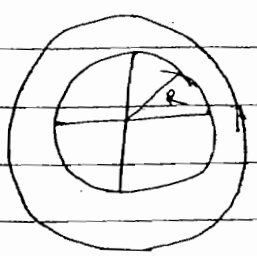
$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}}\, dz$

Let $C = |z| = r > R$

Now, $|a_n| \leq \frac{1}{2\pi} \int \frac{|f(z)|}{|z|^{n+1}} |dz|$

$\leq \frac{1}{2\pi} \int \frac{K|z|^{\alpha}}{|z|^{n+1}} |dz|$.

$= \frac{1}{2\pi} \frac{K}{|z|^{n+1-\alpha}} \int |dz|$



if $n > \alpha$

$$|a_n| \leq \frac{K}{r^{n-\alpha}} < \varepsilon$$

$r \to \infty$

$a_n = 0, \quad n > \alpha$

It's called "extension of Liouville's Thm."

Eg. $|f(z)| \leq K|z|^{\alpha}$

& $\alpha = 5/2$

then $\lceil \alpha \rceil = 2$

$f(z) = a_0 + a_1 z + a_2 z^2$

* $|f(z)| \leq K|z|^{\alpha}$

if $\alpha < 1, 0$

then $f(z)$ is constant.

Corollary :→

* f is entire

& $|f(z)| \leq K|z|^{\alpha}, \quad \alpha > 0$

$\forall z \in \mathbb{C}$

Now, define

$$h(z) = \frac{f(z)}{z^{\alpha}}, \quad z \neq 0$$

But $|h(z)| \leq K \quad \forall z \in \mathbb{C} - \{0\}$

& h is analytic in $\mathbb{C} - \{0\}$

at $z = 0$, h has R.S.

or h is regular.

Let h is defined at 0 as h is analytic in $\mathbb{C}$

$\therefore h$ is constant.

$f(z) = c \cdot z^{\alpha}$

α Should be integer.

* Meromorphism function :→ f is said to be meromorphic at any $z_0 \in \mathbb{C}$, either z_0 is regular pt of f or z_0 is non essential sing.

Note :→ 1) Meromorphism fun^c can have essential sing. at ∞
2) Entire fun^c are mero. not conversly.

Exp. In $\sin\frac{1}{z}$, at 0 essential sing.

So $f = \sin\frac{1}{z}$ is not meromorphism

(3) If f and g are entire then $\frac{f}{g}$ is meromorphic.

4) Tanz is meromorphic

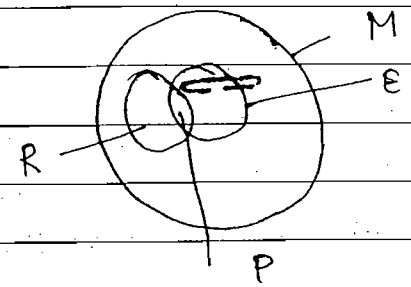
* Rational function :→ A fun^c is said to be rational at any point in the extended complex plane either f has removable is regular & or has non-essential sing. (not essential at ∞)

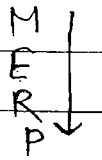
Exp. Sinz is not rational.
Tanz is not rational.

Note :→ 1) Every rational fun^c is meromorphic not conversly.
2) f is rational iff it is of the form $\frac{p(z)}{q(z)}$

where p(z), q(z) are polynomial.

M > E > R > P (Decreasing order)





Result:- For any non constant fun^c (mero.) in any bdd region the no. of zero's or no. of poles should be finite

* $$f(z) = \sum a_n z^n + \sum_{n=1}^{m} b_n \bar{z}^n$$

$$f(1/z) = \sum \frac{a_n}{z^n} + \sum b_n z^n$$

$\exists\ K \in \mathbb{N}$ s.t. $a_n = 0,\ n \geq K$

$$\sum_{n=0}^{K} a_n z^n + \sum_{n=1}^{m} b_n z^n = \frac{P(z)}{q(z)}$$

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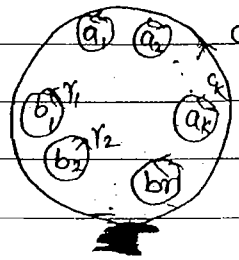
Argument Thm :→

Let a_i = zeros of order n_i
& b_i = pole of order p_i
then $\exists\ \delta_i > 0$ & $\delta_i' > 0$
$g_i(z)$ & $\phi_i(z)$ s.t.



$$f(z) = (z-a_i)^{n_i} g_i(z) \qquad z \in |z-a_i| < \delta_i$$

$$\frac{f'(z)}{f(z)} = \frac{n_i}{z-a_i} + \frac{g_i'(z)}{g_i(z)} \qquad \text{(I)}$$

$$\frac{f'(z)}{f(z)} = -\frac{p_i}{z-b_i} + \frac{\phi_i'(z)}{\phi_i(z)} \qquad \text{(II)}$$

$z \in |z-b_i| < \delta_i'$

if $c_i = |z - a_i| = r_i < \delta_i$

& $\gamma_i = |z - b_i| = r_i' < \delta_i'$

$$\Rightarrow \frac{1}{2\pi i}\int_{C_i} \frac{f'(z)}{f(z)}\,dz = n_i \cdot \frac{1}{2\pi i}\int_{C_i} \frac{1}{z-a_i}\,dz + \frac{1}{2\pi i}\int \frac{g_i'(z)}{g_i(z)}\,dz$$

$$= n_i$$

Similarly,

$$\frac{1}{2\pi i}\int_{\gamma_i} \frac{f'(z)}{f(z)}\,dz = -p_i$$

$$\Rightarrow \frac{1}{2\pi i}\int_{C} \frac{f'(z)}{f(z)}\,dz = \sum \frac{1}{2\pi i}\int_{\gamma_i} \frac{f'(z)}{f(z)} + \sum \frac{1}{2\pi i}\int_{C_i} \frac{f'(z)}{f(z)}$$

$$= \sum n_i + \sum (-p_i)$$

$$= \sum n_i - \sum p_i$$

$$= N - P$$

may be -ve, +ve & zero

$$\Rightarrow \boxed{\frac{1}{2\pi i}\int_{C} \frac{f'(z)}{f(z)}\,dz = N - P}$$

Corollary :→

$$\boxed{\frac{1}{2\pi i}\int_{C} \frac{\psi f'(z)}{f(z)}\,dz = \sum n_i \psi(a_i) - \sum p_i \psi(b_i)}$$

Exp. $\int_{|z|=7.5} z^2 \cot \pi z\,dz$

$$= \frac{1}{\pi}\int_{|z|=7.5} z^2 \frac{\pi \cos \pi z}{\sin \pi z}$$

(In form $\int \frac{\psi(z) f'(z)}{f(z)}$)

Here $\psi(z) = z^2$

$a_i = \pm i^\circ$, $i^\circ = 0$ to 7

$n_i = 1$ (order = 1)

$= \frac{1}{\pi} \times 2\pi i \times 2 \left(a_0^2 + a_1^2 + \cdots a_7^2 \right)$

$= 4i \left(1^2 + 2^2 + \cdots 7^2 \right)$

Exp. $P(z) = a_0 + a_1 z + \cdots a_{n-1} z^{n-1} + a_n z^n$

Let C contains all the zeros of $P(z)$ within it but not on it.

$$\frac{1}{2\pi i} \int_C z \cdot \frac{P'(z)}{P(z)} \, dz = \alpha_1 + \cdots \alpha_n = -\frac{a_{n-1}}{a_n}$$

$$\frac{1}{2\pi i} \int_C z^2 \frac{P'(z)}{P(z)} \, dz = \alpha_1^2 + \cdots + \alpha_n^2$$

$$= \sum (\alpha_i)^2 - 2 \sum \alpha_i \alpha_j$$

$$= \frac{(a_{n-1})^2}{a_n^2} - 2 \frac{a_{n-2}}{a_n}$$

$$= \frac{a^2_{n-1} - 2 a_{n-2} \cdot a_n}{a_n^2}.$$

Q. Let c be a simple closed curve containing all the zeros of $P(z) = z^3 - z - 1$

then find $\frac{1}{2\pi i}\int_c z^9 \frac{P'(z)}{P(z)}\,dz = ?$

Soln:-

$$z^3 - z - 1 = 0 \qquad (\text{term of } z^2 = 0)$$

$$\Sigma\alpha = 0, \quad \Sigma\alpha\beta = -1$$

$$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$$

$$= 0 + 2$$

$$= 2$$

$$\alpha^3 = \alpha + 1$$

$$\Sigma\alpha^3 = \Sigma\alpha + 3 = 3$$

$$(\alpha^3)^3 = (\alpha+1)^3$$

$$\alpha^9 = \alpha^3 + 1 + 3\alpha(\alpha+1)$$

$$\alpha^9 = \alpha^3 + 3\alpha^2 + 3\alpha + 1$$

$$\Sigma\alpha^9 = \Sigma\alpha^3 + 3\Sigma\alpha^2 + 3\Sigma\alpha + 3$$

$$= 3 + 6 + 0 + 3$$

$$= 12$$

if α^3 given then we can easily find

$$\Sigma\alpha^4 = \Sigma\alpha^2 + \Sigma\alpha$$

$$\Sigma\alpha^5 = \Sigma\alpha^3 + \Sigma\alpha^2$$

$$\Rightarrow \frac{1}{2\pi i}\int_C z^9 \frac{P'(z)}{P(z)}\,dz = \alpha^9 + \beta^9 + \gamma^9$$

$$= 12 \text{ //.}$$

Exp. $$f(z) = \frac{(z^2+1)^2 (z^2-1)^3}{(2z-1)^3 (2z+1)^4}$$

a) $C : |z| = 2$

b) $C : |z| = 3/4$

Zeros are :- $z = \pm i$ of order 2

$z = \pm 1$ of order 3

Poles are :- $z = 1/2$ of order 3

$z = -1/2$ of order 4

$$\frac{1}{2\pi i} \int_{C_1} \frac{f'(z)}{f(z)} dz = 10 - 7 = 3$$

$$\int_{C_2} = -7$$

($\because -i = 2$, $+i = 2$, $z = 1, = 3$, $z = -1 = 3$)

(in $|z| = \frac{3}{2}$ zeros are outside the circle)

* Argument principal :→ If f is analytic with in and on a closed curve C and no zero's lies on C then if N is no. of zero's of f with in C counting multiplicity then N is given by —

$$N = \frac{1}{2\pi} \Delta_C \arg f(z)$$

Where $\Delta_C \arg f(z)$ ⟹ The change in the arg $f(z)$ as z move on C in z-plane.

Here C is anticlock wise.

$$\int_C \left(\frac{d}{dz} \log f(z)\right) dz = \log(z) + i \arg f(z)$$

$$= 0 + \Delta_C \, i \arg f(z)$$

$$\Rightarrow \int_C \frac{f'(z)}{f(z)} dz = \Delta_C \, i \arg f(z)$$

$$2\pi i N = \Delta_C \, i \arg f(z)$$

$$\boxed{N = \frac{1}{2\pi} \Delta_C \arg f(z)}$$

Q. $w = f(z) = (z-1)^3 (2z-1)^5 (z-7)^4$

$$\arg w = 3 \arg \underset{w_1}{(z-1)} + 5 \arg \underset{w_2}{(2z-1)} + 4 \arg \underset{w_3}{(z-7)}$$

$C : |z| = 5$

(w_3 not lies on $|z|=5$

$w_3 = z - 7$

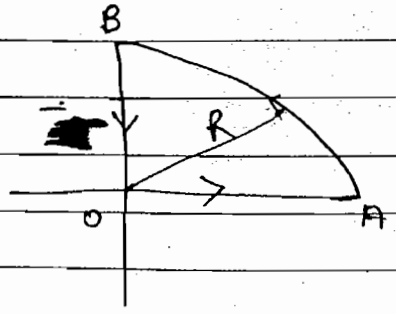
$|w_3 - 7| = |z| = 5$ ✗

$w = f(z) = (z-1)^3 (2z-1)^5 (z-7)$

$\arg f(z) = 3 \times 2\pi + 5 \times 2\pi = 16\pi$

No. of Zero's $N = \frac{1}{2\pi} \times 16\pi = 8$.

Q. $P(z) = z^4 + z + 1$



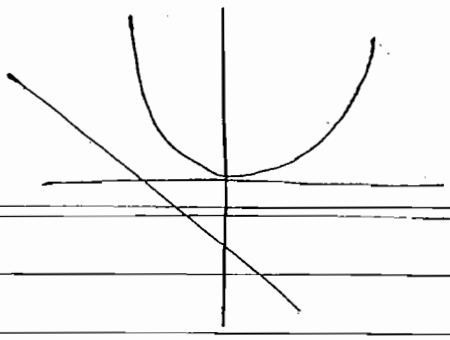
⇒ On X-axis :- $y = 0$

$\therefore P(z) = x^4 + x + 1$

$P(z) = 0$

$x^4 + x + 1 = 0$

$\Rightarrow x^4 = -x - 1$



It has no zero's on X-axis.

On Y-axis :→

$P(z) = (1+y^4) + iy$

P(z) has no zero in Y-axis.

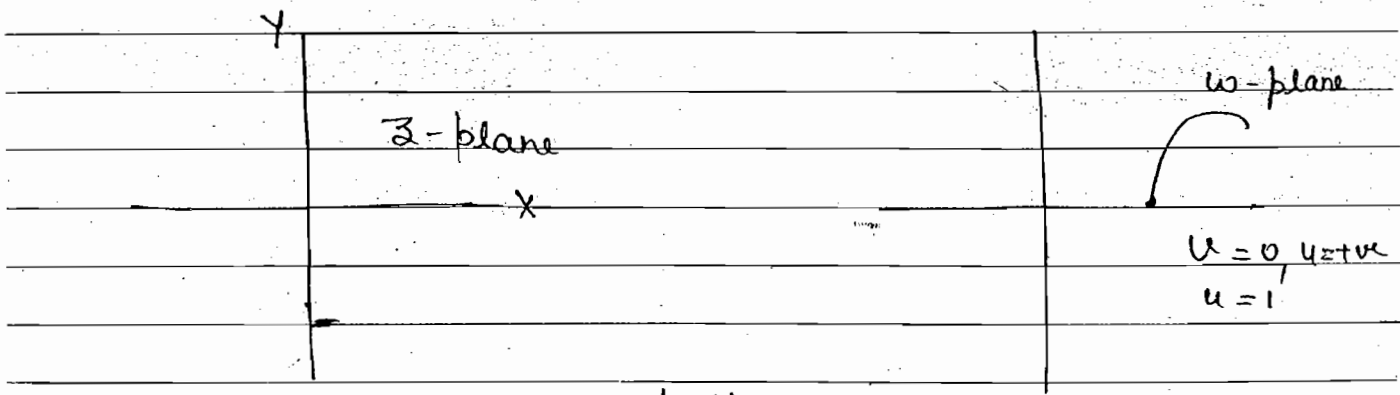
Now, Δc along OA : θ $\quad u = x^4+x+1, \quad v=0$

$$\arg f(z) = \tan^{-1}\frac{v}{u} = 0$$

Δc along arc AB : $z = Re^{i\theta}$

$$P(z) = R^4 e^{4i\theta} + Re^{i\theta} + 1$$
$$= R^4 e^{4i\theta}\left(1 + \frac{1}{R^3}e^{-3i\theta} + \frac{1}{R^4}e^{-4i\theta}\right)$$

$w = (1+y^4) + iy$ $\quad$ i.e. $\psi e^{i\phi}$ $\quad$ where $\phi = 4\theta$



$$\arg f(z) = \tan^{-1}\frac{y}{1+y^4}$$

when $z = \infty$

$$\arg f(z) = \tan^{-1}\lim_{y\to\infty}\frac{y}{1+y^4}$$

∴ Change in arg = 0

P(z) has only one zero in 1st quadrant

$$\frac{1}{2\pi}(\text{Change in } \Delta\arg) = \frac{1}{2\pi}\times 2\pi = 1 \text{ //}$$

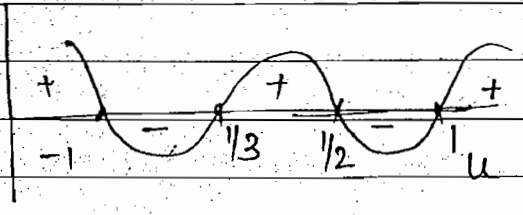
This Exp. P(3) has zero in diff. quadrant:

* On y-axis

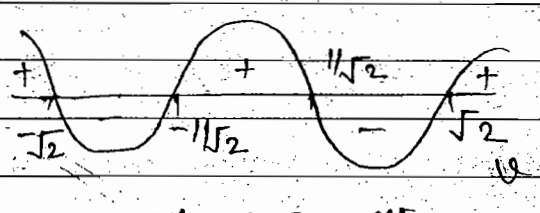
$$f(3) = (2y-1)(3y-1)(y^2-1) + i(y^2-2)(2y^2-1)$$

y	∞	$\infty \to \sqrt{2}$	$\sqrt{2}$	$\sqrt{2}-1$	1	$1-1/\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}-1/2$	1/2
u	+	+	+	+	0	−	−	0	
v	+	+	0	−	−	−	0	+	
Arg f(3)	$\tan^{-1}\alpha$								

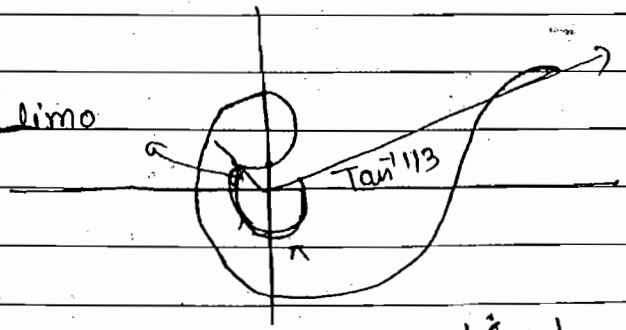
$2(y-1/2)(y-1/3)(y+1)(y+1)$



$(y-\sqrt{2})(y+\sqrt{2})(y-1/\sqrt{2})(y+1/\sqrt{2})$



along $R = \frac{n\pi}{2}$



arg at ∞ = $\tan^{-1}\left(\frac{1}{3}\right)$

at $\lim \infty = \tan^{-1}\left(\frac{1}{3}\right)$

Then add all three angle

(clockwise = -ve sign)

$$-\left[\pi + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}(1)\right]$$

$$\Rightarrow \tan^{-1}\frac{(y^2-2)(2y^2-1)}{(2y-1)(3y-1)(y^2-1)}$$

$$= \tan^{-1}\frac{2(y^2-2)\,2y^2\left(1-\frac{1}{2y^2}\right)\left(1-\frac{2}{y^2}\right)\cdot y^2}{2y\left(1-\frac{1}{2y}\right)\cdot 3y\left(1-\frac{1}{3y}\right)y^2\left(1-\frac{1}{y^2}\right)}$$

$$\lim_{n\to\infty} \tan^{-1}\frac{1}{3} \;//$$

ROUCHE'S THM :→ Let C be a simple closed curve, f & g are analytic with in & on C i.e. $|g(z)| < |f(z)|$

Then f and f+g will have same no. of zero's with in C counting multiplicity.

OR $|f-g|$ & $|g-f| < |f|$

$|g(z) - f(z)| < |f(z)|$ on C.

f & g have same no. of zero's within C.

$f+g \quad \leftarrow N_1 = \frac{1}{2\pi} \Delta_C \arg (f+g)$

$= \frac{1}{2\pi} \Delta_C \arg f + \frac{1}{2\pi} \Delta_C \arg \left(1 + \frac{g}{f}\right)$

$N_2 = \frac{1}{2\pi} \Delta_C \arg f$

$N_1 - N_2 = \frac{1}{2\pi} \Delta_C \arg \left(1 + \frac{g}{f}\right)$

Let $w = 1 + g/f$

$\Rightarrow w - 1 = g/f$

on C $|w-1| = \left|\frac{g}{f}\right| < 1$

$\Delta_C \arg w = 0$

$N_1 - N_2 = 0$

$\Rightarrow \boxed{N_1 = N_2}$

Exp. $p(z) = z^4 + z^3 + 1$

$|z| = \frac{1}{2}$

$f(z) = 1$

$g(z) = z^4 + z^3$

$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{z^4+z^3}{1}\right| \leq |z|^4 + |z|^3$

$\leq \frac{1}{16} + \frac{1}{8} < 1$

$|g(z)| < |f(z)|$

$f(z)$ has no zeros.

$|z| = 2$:— $f(z) = z^4$

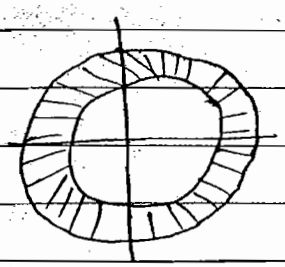
$g(z) = z^3 + 1$

$\left|\frac{g(z)}{f(z)}\right| < \frac{|z|^3+1}{|z|^4} = \frac{9}{16} < 1$

$f + g$ has same zero's.

f has 4 zero's.

$\frac{1}{2} < |z| < 2$



Exp. $p(z) = z^5 - 7z^3 + 2z^2 + 1$

$|z| = 1$

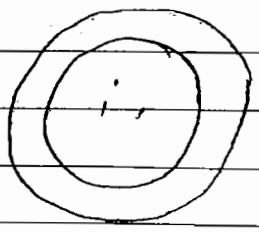
$f(z) = -7z^3$

$g(z) = z^5 + 2z^2 + 1$

$\left|\frac{g(z)}{f(z)}\right| < \frac{|z|^5 + 2|z|^2 + 1}{7 \cdot |z|^3}$ $\frac{4}{7} < 1$

f has no. of zero's = 3

$1 < |z| < 2$



$|z| = 3$

$f(z) = z^5$

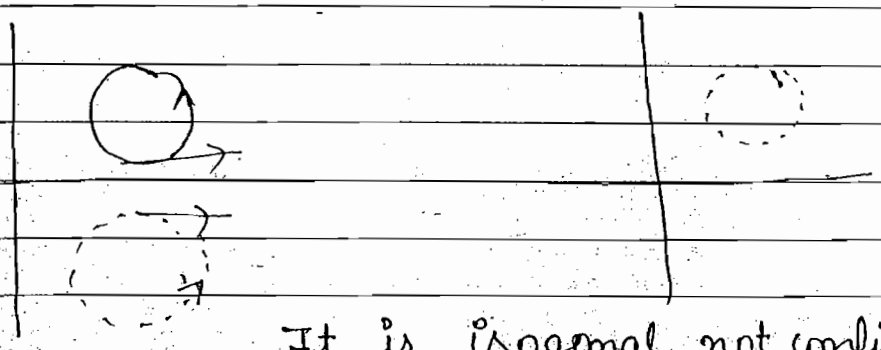
$g(z) = -7z^3 + 2z^2 + 1$

PHASE-3

CONFORMAL MAPPING :→ Let f is defined on D and $z_0 \in D$

f is Said to conformal at z_0 if the angle b/w any two oriented curves is preserve at z_0 under f in magnitude & sense both.

If only magnitude is preserve not sense is called "isogonal" at that point.

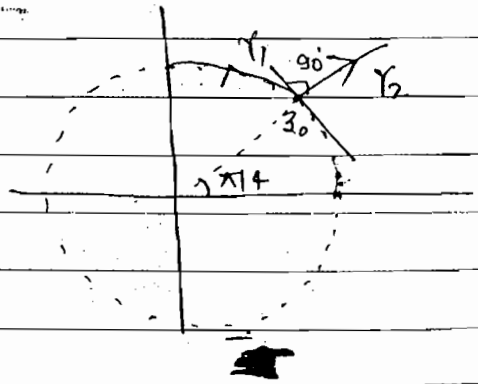


It is isogonal not confirmal.

Exp $\gamma_1 = e^{it}, \quad t \in [\pi/4, \pi/2]$

$\gamma_2 = te^{i\pi/4}, \quad t \in [1,3]$

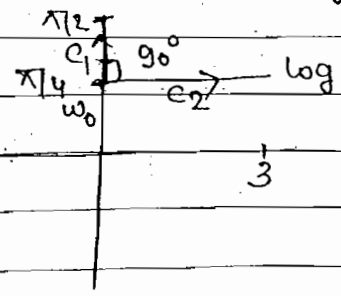
$z_0 = e^{i\pi/4} = \frac{1}{\sqrt{2}} + i\cdot\frac{1}{\sqrt{2}}$



$w = \log z$

$c_1(t) = w(t) = \log e^{it} = it + 0 \quad t \in [\pi/4, \pi/2]$

$c_2(t) = \log t + i\pi/4, \quad t \in [1,3]$



Hence $\log z$ is conformal mapp. at $z = e^{i\pi/4}$

Let f is diff. at z_0 & $f'(z_0) \neq 0$
Let γ is an oriented curve at z_0.
then the image of γ under f is given by C.

$$C(t) = (f \circ \gamma)(t) = f(\gamma(t))$$

$\gamma(t)$ & $\text{Arg}\ \gamma'(t) = \theta$

$$C(t) = (f \circ \gamma)(t)$$

$$\text{Arg}\ C'(t) = \phi$$

$\gamma(t)$ is mapped into $C(t)$
by $w = f(z)$ & $\arg \gamma'(t) = \theta$
then $\arg C'(t) = \phi$

$$C'(t) = f'(\gamma(t)) \cdot \gamma'(t)$$

$$\text{Arg}\ C'(t_0) = \text{Arg}\ f'(\gamma(t_0)) + \text{Arg}\ \gamma'(t_0)$$

$$\Rightarrow \boxed{\phi = \text{Arg}\ f'(z_0) + \theta} \qquad f'(z) \neq 0$$

$\psi_0 = \text{Arg}\ f'(z_0)$
- angle of rotation.

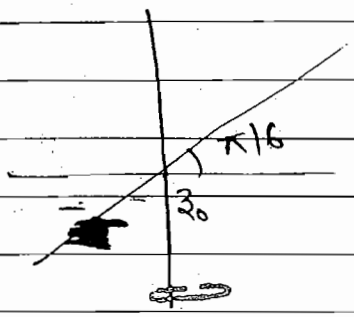
Q → $f(z) = (1+i)e^z$

$z_0 = 0$

$f'(z_0) = (1+i)$

$\text{Arg}\ f'(z_0) = \frac{\pi}{4}$

$$\boxed{\phi = \frac{\pi}{4} + \frac{\pi}{6}}$$



Conformal on D: →
If f is conformal at every point of D then we say f is conformal on D.

Result :→ 1) If f is analytic in D then f is conformal at any pt. $z_0 \in D$ iff $f'(z_0)$ is non zero.

2) If f is analytic in D and at z_0, f' has zero of order k then under f the angle b/w any two oriented curve is multiplied by $k+1$.

$$\boxed{(\phi_1 - \phi_2) = (k+1)(\theta_1 - \theta_2)}$$

$$* \quad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$f'(z_0) \cdots f^k(z_0) = 0, \quad f^{k+1}(z_0) \neq 0$$

$$f(z) = f(z_0) + 0 + \cdots 0 + a_{k+1}(z-z_0)^{k+1} + \cdots$$

$$f(z) - f(z_0) = (z-z_0)^{k+1} \left[a_{k+1} + a_{k+2}(z-z_0) + \cdots \right]$$

$$f(z) - f(z_0) = (z-z_0)^{k+1} \cdot g(z)$$

$g(z)$ is analytic & $g(z_0) \neq 0$

$$\operatorname{Arg}(f(z) - f(z_0)) = (k+1)\operatorname{Arg}(z-z_0) + \operatorname{Arg} g(z)$$

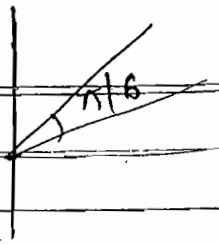
Along γ :→

$$\boxed{\phi = (k+1)\theta + \operatorname{Arg} g(z_0)}$$

$$\phi_1 = (k+1)\theta_1 + \alpha$$

$$\phi_2 = (k+1)\theta_2 + \alpha$$

$$\boxed{\phi_1 - \phi_2 = (k+1)(\theta_1 - \theta_2)}$$



Q. $f(z) = 2z^3 + 1$

$f'(z) = 6z^2 \Rightarrow f'(z) = 0$

$f''(z) = 12z \quad f''(z) = 0$

$f'''(z) = 12 \quad f'''(0) \neq 0$

has order 2

$K = 2, \quad \phi = (K+1)\theta$

$= 3\pi/6$

$|z| = 1$

$\gamma_1(t) = e^{it}$

$\gamma_2(t) = t$

$f(z) = 2z + z^2$

$f'(z) = 2 + 2z = 0$

$z = -1$

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$f : D \longrightarrow D$

$w = f(z)$

Q. $f(z) = z^2 = x^2 - y^2 + 2xyi$

$u = x^2 - y^2$

$v = 2xy \qquad y = x - 1$

$u = x^2 - \frac{v^2}{4x^2}$

$\Rightarrow y = \frac{2x\, v}{2x}$

$2x(x-1) = v \quad \Rightarrow 2x = \frac{v}{(x-1)}$

$u = x^2 - \frac{4x^2(x-1)^2}{4x^2} \qquad u = \frac{x v^2}{x 4(x-1)^2} - \frac{v^2}{4x^2}$

$= 1 - 2x$

$u = x^2 - y^2$

$u = x^2 - (x-1)^2$

$u = 2x - 1$

$2x = u + 1$

$v = 2x(x+1)$

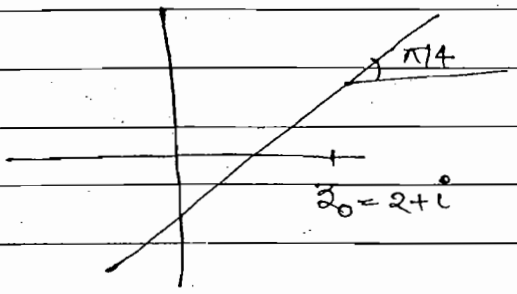
$v = 2\left(\frac{u+1}{2}\right)^2 - (u+1)$

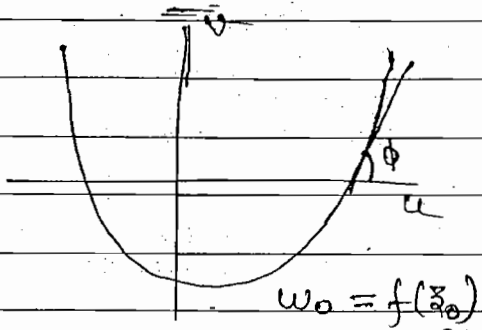
$2x = u+1 \Rightarrow x = \frac{u+1}{2}$

$v = 2\left(\frac{u+1}{2}\right)^2 - (u+1)$

$\boxed{u^2 = 2(v + 1/2)}$

$u^2 = 2\left(v + \frac{1}{2}\right)$





$w_0 = f(z_0)$

$w_0 = 3 + 4i$

$f(z) = z^2$

$f'(z) = 2z = 2(2+i)$

$f'(z_0) = 4 + 2i$

$\text{Arg } f'(z_0) = \text{Tan}^{-1}\left(\frac{1}{2}\right)$

$\phi = \text{Tan}^{-1} 1 + \text{Tan}^{-1} \frac{1}{2}$

$= \text{Tan}^{-1} \frac{1 + 1/2}{1 - 1/2} = \text{Tan}^{-1} 3$

Or $u^2 = 2(v + 1/2)$

$2u = \frac{2\,dv}{du} \Rightarrow \frac{dv}{du} = u$

$\frac{dv}{du} = 3$

$\boxed{\phi = \text{Tan}^{-1} 3}$

Q. $f(z) = \frac{1}{z} = \frac{\bar{z}}{|z|^2}$

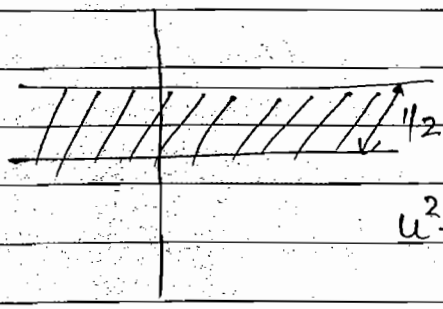
$f(z) = \frac{x}{x^2+y^2} + \frac{(-y)}{x^2+y^2}$

$x^2 + y^2 = 1$

$-ve$ (so below x-axis)

$u = x \quad , \quad v = -y$

$u^2 + v^2 = 1$ ⟂



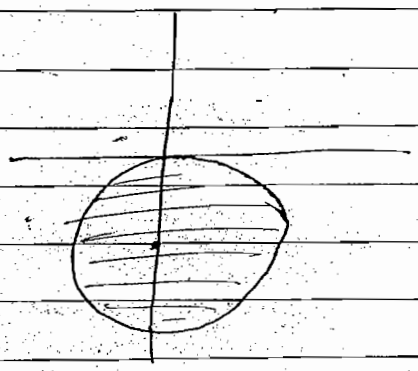
$0 < y < \frac{1}{2}$

$u^2 + v^2 = \frac{1}{x^2+y^2} = \frac{-v}{y} < -2v$

$\frac{v}{y} > \frac{v}{1/2} > 2v$

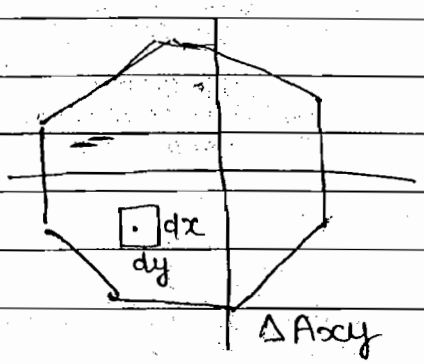
$\frac{-v}{y} < -2v$

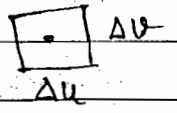
$u^2 + v^2 + 2v < 0$



* $f : \mathbb{R}^2 \to \mathbb{R}^2$

$f(x,y) = (u,v)$





$\lim_{\Delta Axy \to 0} \frac{\Delta Auv}{\Delta Axy} = J$

$\iint du\,dv = \iint J\,dx\,dy$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} u_x & -v_x \\ v_x & u_x \end{vmatrix}$$

Let $u+iv = f(z)$ is analytic

$\Rightarrow u_x = v_y$ & $u_y = -v_x$

$$\Rightarrow \begin{vmatrix} u_x & -v_x \\ v_x & u_x \end{vmatrix} = u_x^2 + v_x^2 = |u_x + iv_x|^2 = |f'(z)|^2$$

$f : D \longrightarrow D$

$f(z) = u+iv$ analytic in D

$|f'(z_0)|^2 = J$ at z_0

is called magnification factor.

& $\boxed{\text{Arg } f'(z_0) = \text{Angle of Rotation}}$

if $0 < |f'(z_0)|^2 < 1$ - ve magnifaction

$|f'(z_0)|^2 > 1$ + ve " "

if $J = 1$ (No change in area)

where

$|f'(z_0)|$ = Scalar factor.

Q. (i) $f(z) = (1+i)z + e^{i\pi/4}$

(ii) $f(z) = 1/z$

(i) $f'(z) = (1+i)$

$|f'(z)| = \sqrt{2}$

$J = 2$

(ii) $f'(z) = -1/z^2$

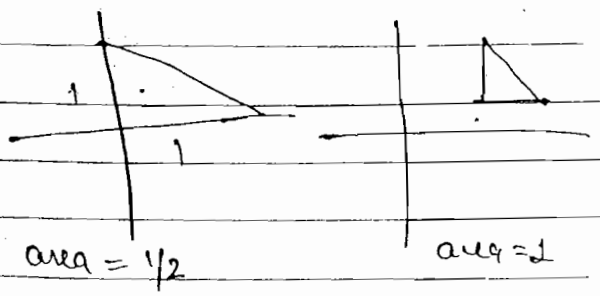
$|f'(z)| = \frac{1}{x^2+y^2}$

$J = \frac{1}{(x^2+y^2)^2}$

In this case $\Rightarrow$, Scalar factor all are constant.

Area of $w(z) = 2$ area $f(z)$

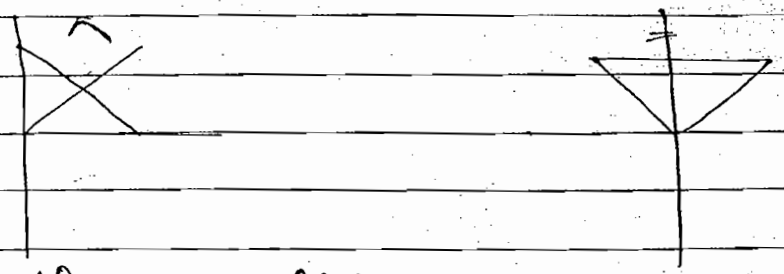
$\because J = 2$



Some basic Transformation :→

1) Translation :→ i. $f(z) = z + z_0$

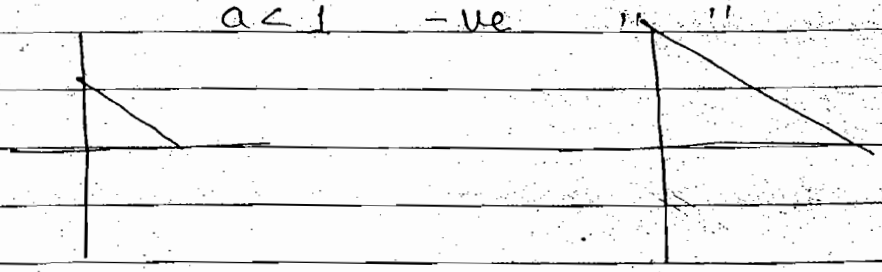
2) Rotation :→ $f(z) = e^{i\theta} z$ $\theta \in \mathbb{R}$



3) Magnification :→ $f(z) = az$, $a \in (0, \infty)$

$a > 1$ +ve magnifaction

$a < 1$ -ve " "



4) Inversion :→

5) Möbious Transformation | Bilinear transformation | Linear trans fractimal :→

$$f(z) = \frac{az+b}{cz+d} \quad , \quad ad - bc \neq 0$$

$$f : \mathbb{C} - \{-d/c\} \longrightarrow \mathbb{C}$$

$$= D$$

Property :-

f is analytic in Domain $\mathbb{C} - \{-d/c\}$
and $\{-\frac{d}{c}\}$ is simple pole.

2) $f: \mathbb{C}_\infty \to \mathbb{C}_\infty$

$$f(z) = \begin{cases} \frac{az+b}{cz+d} & z \neq -d/c \\ \infty & z = -d/c \\ a/c & z = \infty \end{cases}$$

3) Over $\mathbb{C}_\infty$, möbius transformation is one-one & onto i.e. invertible.

(b) Identity fun^c is also bilinear transfo.

$a = 1, \; b = 0 = c, \; d = 1$

$ad - bc \neq 0 \quad \therefore$ bilinear.

(c) Composition of möbius transformation is also a möbius trans.

(d) Set of all ~~m~~ linear trans. (möbius trans.) on $\mathbb{C}_\infty$ form group w.r. to composition of mapping as binary operation which is infinite non-abelian gp.

(e) Sum, product of B.T. is not B.T.

Exp. $\phi: GL(2, \mathbb{C}) \to M(\mathbb{C}_\infty)$

$$\phi(A) = \frac{az+b}{cz+d}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Every gp. of $GL(2, \mathbb{C})$ represent möbius transfo.

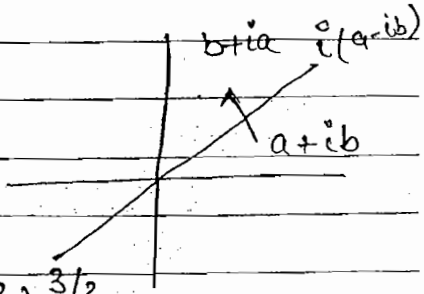
$\because (ad - bc \neq 0)$

Symmetric point | Inverse point | conjugate point :-

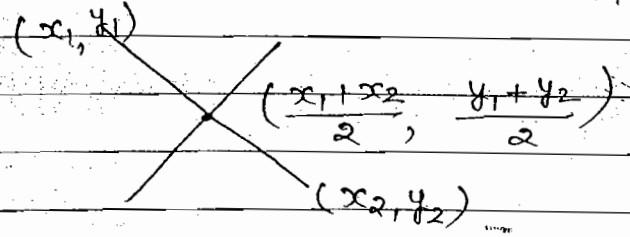
$z_1, z_2 \in C_\infty$ are said to be inverse point w.r. to curve c if they represent image of each other taking curve c as mirror.

$\Rightarrow$ which point lies on the curve are called self inverse | self conjugate point.

c_1	x-axis	$z_2 = \bar{z}_1$
c_2	y-axis	$z_1 = -\bar{z}_1$
c_3	y=x axis	$z_2 = i\bar{z}_1$



Radius of curvature $= \dfrac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{d^2y/dx^2}$

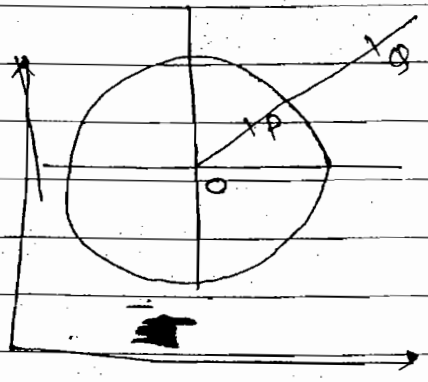


*

$c: |z - z_0| = r$

points P & Q are said to be Symmetric w.r. to circle c if

1) OPQ are colinear.
2) P & Q are on the same side of O.



$$\boxed{OP \cdot OQ = r^2}$$

$$\boxed{z_2 = z_0 + \frac{r^2}{\bar{z}_1 - \bar{z}_0}}$$

Q. Find the inverse of $2+3i$

$|z-i| = 1$

$z_2 = \dfrac{i + r^2}{2-3i+i}$

$= \dfrac{i + r^2}{2-2i}$

$OP \cdot OQ = r^2 \Rightarrow 1$

$= i + \dfrac{1}{2-2i} \Rightarrow \dfrac{2i-2i^2+1}{2-2i}$

$= \dfrac{2i+3}{2(1-i)} \dfrac{(1+i)}{(1+i)}$

$z_2 = \dfrac{2i+3-2+3i}{2(1+1)} = \dfrac{5i+1}{4}$

$z_2 =$

* Symmetric, pt for any circle centre is ∞.

Q. $|z|=1$ find the inverse of 2.

$z_2 = 0 + \dfrac{1}{2+0} = \dfrac{1}{2} = \infty$

Fixed point :→ z_0 is said to be a fixed pt $w = f(z)$ if $f(z_0) = z_0$

1) For translation $f(z) = z+z_0$ $z_0 \neq 0$ → then no fixed pt in C. but in C_∞, ∞ is the only fixed pt.

2) Identity trans. $f(z) = z$ has infinite fixed pt. i.e. every element of C_∞ is fixed pt

3) For any möbius transformation

$$f(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0$$

has atmost 2 fixed pt.

$$f(z) = z$$

$$\therefore \frac{az+b}{cz+d} = z$$

$$cz^2 + dz - az - b = 0$$

$$\Rightarrow cz^2 + z(d-a) - b = 0$$

$$z = \frac{-(d-a) \pm \sqrt{(d-a)^2 + 4bc}}{2c}$$

has atmost 2 fixed pt.

Möbius has atmost 2 fixed pt not identity.

*) If for any mobius transformation if $0, 1, \infty$ are fixed pts then $f(z) = z$ i.e. f is identity.

Note :- when a möbius transformatim is only 1 fixed pt in c i.e. $\sqrt{(a-d)^2 + 4bc} = 0$ then Such trans. are called parabolic.

$$\boxed{z = \frac{a-d}{2c}}$$ is fixed point.

* Normal / canonical form of mobius transformations :-
Let p & q are distinct fixed pt of

$$f(z) = \frac{az+b}{cz+d}$$

Then w can be express as —

$$\frac{w-p}{w-q} = k\left(\frac{z-p}{z-q}\right)$$

where K is complex constant.

$$\frac{\frac{az+b}{cz+d} - p}{\frac{az+b}{cz+d} - q} = K\left(\frac{z-p}{z-q}\right)$$

$$\boxed{K = \frac{a-cp}{a-cq}}$$

This is called normal form of möbius trans. which has 2 fixed pt.

Q. $f(z) = \frac{1-z}{1+z}$

$f(-z) = \frac{1+z}{1-z} = -\frac{1}{f(z)}$

It has 2 fixed point

$$\frac{\frac{1-z}{1+z} - p}{\frac{1-z}{1+z} - q} = K\left(\frac{z-p}{z-q}\right)$$

$$K = \frac{1-p}{1-q}$$

fixed point $\frac{1-z}{1+z} = z$

$1 - z = z + z^2$

$z^2 + 2z - 1 = 0$

$z = \{1, -1\}$

$z = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm 2\sqrt{2}}{2}$

$z = -1 \pm \sqrt{2}$

$K = \frac{(-1) - (-1+\sqrt{2})}{(-1) - (-1-\sqrt{2})} = (-1)$ elliptic

* Classification of M.T. when two distinct fixed pt →

1) Elliptic → if $k \neq 1$
but $|k| = 1$

2) Hyperbolic → when k is +ve real no.

If No elliptic & Hyperbolic then loxodromic

*

$$f(z) = \frac{z}{1-z}$$

fixed → $f(z) = z$

$z = z(1-z)$

$z - z - z^2 \Rightarrow z = 0, 0$

Only one fixed point 0

∴ parabolic.

Note:- when only one fixed pt then the normal form is given by

$$\boxed{\frac{1}{w-p} = k + \frac{1}{z-p}} \qquad \boxed{k = \frac{c}{a-cp} = \frac{c}{a+d}}$$

In above Exp., $\frac{1}{w-0} = k + \frac{1}{z-0}$

$\frac{1}{w} = k + \frac{1}{z}$, $\Rightarrow \boxed{k = \frac{a-cp}{a-cq}}$

$k = \boxed{\frac{1}{w} = -1 + \frac{1}{z}}$.

Properties :→ A mobius trans. maps circle into circle only.

Here line is particular case of circle of radius ∞.

) Under möbius trans. if curve γ is mapped into curve C then the point symm. w.r. to γ will be map into points of w.r. to C.

Symmetric.

$$f : D \to D$$
$$|z| \le 1 \qquad |w| \le 1$$
$$f(z) = \frac{az+b}{cz+b}$$

Case – I :→ $c = 0 \Leftrightarrow \infty$ correspond to ∞.

$$f(z) = Az + B \qquad \left(|A| = 1,\ B = 0 \right)$$
$$\Rightarrow f(z) = e^{i\lambda} z \qquad \left(\because |w| \le 1 \right)$$

Case – II :- when $c \neq 0$ then $f(z) = \frac{a}{c}\left(\frac{z + b/a}{z + d/c}\right)$

$z = -b/a \Rightarrow w = 0$

$z = -d/c \Rightarrow w = \infty$

As 0 & ∞ are symmetric w.r. to $|w| \le 1$

$\Rightarrow -b/a$ & $-d/c$ " " " " $|z| = 1$

ie. if $-\frac{b}{a} = \alpha \Rightarrow -\frac{d}{c} = \frac{1}{\bar{\alpha}}$

$$f(z) = \left(\frac{a\bar{\alpha}}{c}\right)\left(\frac{z-\alpha}{\bar{\alpha}z - 1}\right)$$

$$f(1) = \left(\frac{a\bar{\alpha}}{c}\right)\left(\frac{1-\alpha}{\bar{\alpha} - 1}\right)$$

as $1 \in |z| = 1$

$f(1) \in |w| = 1$

$\Rightarrow |f(1)| = 1$

$\Rightarrow \left|\frac{a\bar{z}}{c}\right| \left|\frac{1-\alpha}{\bar{\alpha}-1}\right| = 1$

$\Rightarrow \left|\frac{a\bar{z}}{c}\right| = 1$

$\frac{a\bar{z}}{c} = e^{i\theta}$

$$\boxed{w = e^{i\theta}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)}$$

This Transformation is 1-1, onto from unit disk to unit disk.

Take $\alpha = 0$

$$\phi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$$

$\Rightarrow \phi_\alpha : |z| \le 1 \to |w| \le 1$

$\boxed{\phi_\alpha(\alpha) = 0}$

$\phi(\alpha)$ of $\phi(\alpha)$ is identity map.

* $f(1/2) = 0$

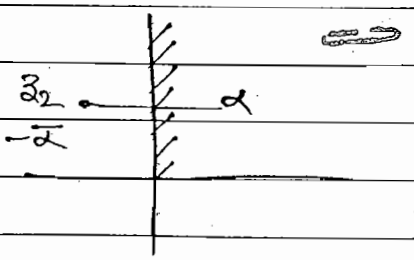
$g = f \circ \phi_{1/2}$

$g(z) = f(\phi_{1/2}(z))$

$g(0) = f(1/2) = 0$

* $-b/a = \alpha$

then $-d/c = -\bar{\alpha}$



$$w = \frac{1+z}{1-z}$$

$$w+1 = \frac{1+z+1-z}{1-z}$$

$$\frac{1-z}{2} = ?$$

Cross Ratio :→ Let z_1, z_2, z_3, z_4 are distinct complex no. then the four tupples defined as –

$$(z_1, z_2, z_3, z_4) = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)} \in \mathbb{C}$$

There are 6 atmost possible cross ratio of 4 distinct Complex no.

Result :- Under möbius transformation cross ratio is preserve.

i.e. $$\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_3-z_1)}$$

$$z_i \longrightarrow w_i$$

Under möbius tran. image 3 distinct complex no. given then trans. known completely.

Hence if z_1, z_2, z_3 are mapped to w_1, w_2, w_3 respectively then –

Let $z_4 = z$, mapped to $w_4 = w$

$$\frac{(z_1-z_2)(z_3-z)}{(z_2-z_3)(z-z_1)} = \frac{(w_1-w_2)(w_3-w)}{(w_2-w_4)(w-w_1)}$$

if $z_4 \to \infty$ then $-\left(\frac{z_1 - z_2}{z_2 - z_3}\right)$

POWER SERIES

$\sum_{n=0}^{\infty} a_n (z^n - z_0)^n$ is defined as formal power series.

$\Rightarrow \sum_{n=0}^{\infty} a_n z^n$ is power series around zero.

Result & properties :→

1) The largest R so that series converges at every pt of open disk i.e. $|z - z_0| < R$ is defined as Radius of convergence & this disk itself called domain of convergence.

2) If $\sum a_n (z - z_0)^n$ converges at $z = \alpha$ then it converges $\forall\ z \in |z - z_0| < |z_0 - \alpha|$

3) When Roc is 0 then $\sum_{n=0}^{\infty} a_n z^n$ converges at $z = z_0$ only.

4) If R is Roc then power series converges uniformly $|z - z_0| \leq r < R$ & absolutely on Compact region $|z - z_0| < r < R$

5) $\sum a_n (z - z_0)^n = f(z)$
$z \in |z - z_0| < R$
This f is called uniform limit of the power series.

6) The uniform limit of power series is analytic fun^c in domain of convergence then power series is become taylor series of fun^c in nbd of z_0.

7) The radius of convergence of the power series is the distance of nearest sing. of uniformly limit fun^c from the point z_0.

Exp. $\sum a_n z^n = \frac{1}{1-z}$

$R = 1$

$1 - z = 0$

$z = 1$

$1 - \frac{1}{4} = \frac{1}{R}$

z^{2n}

$z = 2$

$z^2 = 2$

$z = \sqrt{2}$

$R = 1$

$1 - z = 0$

$z = 1$

$z = R + 1$

$z - 1 = 0$

$z = 1$

Roc $= 1$

$\sum a_n z^n = 1 + z + z^2 \cdots + z^n \cdots = \frac{1}{1-z}, \quad |z| < 1$

Exp. $a_n = \begin{cases} \frac{1}{2^n}, & n \text{ is even} \\ (-1)^{\frac{n+1}{2}} \frac{1}{\lfloor n} + \frac{1}{2^n}, & n \text{ is odd} \end{cases}$

$\sum a_n z^n = \left(-\frac{1}{\lfloor 3} + \frac{1}{2^3}\right) z^3$

$= -\frac{1}{3!} z^3 + \frac{1}{2^3} z^3 \cdots$

$= \frac{1}{\lfloor 5} z^5 + \frac{1}{2^5} z^5 \cdots$

$\sum a_n z^n = \left(1 - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 + \cdots + \frac{z}{2} + \frac{z}{2^2} + \cdots\right)$

$= \sin z + \frac{z}{2-z}$

Roc $= 2$ //

Exp. find the ROC ?

given $a_0 = 1$, $a_1 = 1$

$a_{n+2} = a_{n+1} + a_n$

Sol^n:- This is Recurrance Eqn^n. (coff. of a_{n+2} & coff. of $z \times a_{n-1}$ coff of $z^2 \times a_n$)

$P(z) = 1 - z - z^2$

$P(z) \cdot f(z) = (1 - z - z^2)(a_0 + a_1 z + a_2 z^2 + \cdots)$

$= a_0 + (a_1 - a_0) z + (a_2 - a_1 - a_0) z^2$

$= a_0 + (a_1 - a_0) z + \sum_{n=2}^{\infty} (a_n - a_{n-1} - a_{n-2}) z^n$

$= 1 + (1-1) z + 0 = 1$

$f(z) = \dfrac{1}{(1 - z - z^2)}$

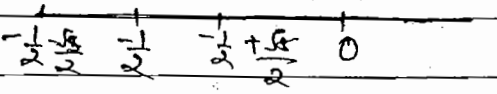
ROC $= 1 - z - z^2 = 0$

$z^2 + z - 1 = 0$

$\dfrac{-1 \pm \sqrt{1+4}}{2} = \dfrac{-1 \pm \sqrt{5}}{2}$

$\left| \dfrac{-1 \pm \sqrt{5}}{2} - 0 \right| = \dfrac{-1 + \sqrt{5}}{2}$ ROC.

if $\sum a_n (z-1)^n$ then



$\left| -\dfrac{1}{2} + \dfrac{\sqrt{5}}{2} - 1 \right| = \dfrac{\sqrt{5}}{2} - \dfrac{3}{2}$

$\left| -\dfrac{1}{2} - \dfrac{\sqrt{5}}{2} - 1 \right| = -\dfrac{\sqrt{5}}{2} - \dfrac{3}{2}$

Min $\{ |z_0 - \alpha_i| : i = 1 \text{ to } k \}$

Had-mard formula :→

define $b_n = \left|\frac{a_{n+1}}{a_n}\right|$ if possible

or $c_n = |a_n|^{1/n}$ always possible.

Then Roc of $\sum a_n z^n$ is given by —

$$\frac{1}{R} = \lim b_n = \lim c_n.$$

Exp. $a_n = n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n+1}{n}\right| = 1 \text{ is Roc.}$$

Exp.
$$a_n = \begin{cases} 1 & n = k! \text{ for some } k \\ 0 & \text{else} \end{cases}$$

$$\text{Roc} = 1 \quad \left(\begin{matrix} k=1,\ 1=1! \\ k=2,\ 2=2! \end{matrix}\right)$$

Exp.
$$\sum z^{n!}$$
$$= z^{0!} + z^{1!} + z^{2!} + \cdots \quad \left(\because \text{coff. of } z^n = 1 \text{ or } 0\right)$$
$$= 2z + z^{2!} + \cdots$$
$$\text{Roc} = 1$$

Exp. $a_n = \prod_{i=1}^{k} (n_i+1)$ for some finite k.

$$c_n = (n_i+1)^{1/n} (n_i+1)^{1/n} \cdots (n_k+1)^{1/k}$$

a_n = no. of divisor of n.

$$p_i^{50} = p_1^{50n_1} \cdot p_2^{50n_2} \cdots$$

Roc = 1

(*) $a_{n+2} = a_{n+1} + \beta a_n$

$\xi^2 - \alpha\xi - \beta = 0$

$\xi = a, \beta b$

$\boxed{a_n = c_1 a^n + c_2 b^n}$

If linear then valid only.

If Non-Linear then –

28/11/2014

$f : D \longrightarrow D$

$f(z) = \sum a_n z^n$

$$a_n = \begin{cases} \frac{1+i}{2} & n \text{ is odd} \\ \frac{1-i}{3} & n \text{ is even} \end{cases}$$

$a_n = |\alpha|^{1/n}, |\beta|^{1/n}$

$n \to \infty \quad \alpha^{1/n} = 1$

$\therefore$ Roc $= 1$

$$b_n = \begin{cases} \left(\frac{1+i}{2}\right)^n & \text{odd} \\ \left(\frac{1-i}{3}\right)^n & \text{even} \end{cases}$$

$|a_n|^{1/n} = \left|\frac{1+i}{2}\right|^{n \times \frac{1}{n}}$

$= \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

odd $\to \frac{\sqrt{2}}{3}$, $\frac{\sqrt{2}}{2} \to$ even

$\text{Sup} = \frac{\sqrt{2}}{2} = \frac{1}{R} \Rightarrow$ Roc $= \sqrt{2}$ //.

$|b_n|^{1/n} = \frac{|1+i|}{2} = \frac{\sqrt{2}}{2}$

$= \frac{|1-i|}{3} = \frac{\sqrt{2}}{3}$

Mean value property :→ Let f is define on a domain D we say f satisfied mean value property at any pt 'a' in D if –

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\, d\theta$$

where $|z-a| < r \subset D$ for some $r > 0$

↳ If f is analytic in D & f has M.V.P. in D
Crauss Mean value Thm.

Thm :- Lemma :→ Let f is continuous with in & on D & satisfied mean value property of D then maximum of $|f|$ is not attainted in D unless constant.

Maximum modulus principal :→ Let f is analytic in D then $|f|$ does not attained its maximum in D unless Constant.

Q :- $f : |z| < 1 \longrightarrow |z| \le 5$

$f(1/2) = 3+4i$

$\Rightarrow$ f is constant $\left(\because |1/2| < |1| \quad \therefore \text{maximum m.p.}\right)$

* **C - class Thm :→**

Let f is analytic in D & continuous on boundary of D (i.e. ∂D) and if for every $z \in \partial D$ $\exists$ $z' \in D$ s.t. $|f(z)| \le |f(z')|$

then f is constant.

Maximum modulus Thm:→ If f is analytic in D, conti. on the boundary of D, then maxi. $|f|$ is attain on the boundary of D but never interior point unless f is constant.

Exp.

$$f(z) = z^2 + 2 \qquad D \cup \partial D = |z| \le 1$$

$$|f(z)| \le |z|^2 + 2 \le 3 \text{ on } \partial D$$

$$|f(z)| = 3$$

$$|f(z)| = |e^{2it} + 2|$$

$$= |(\cos 2t + 2) + i \sin 2t|^{1/2}$$

$$u = |f(z)|^2$$

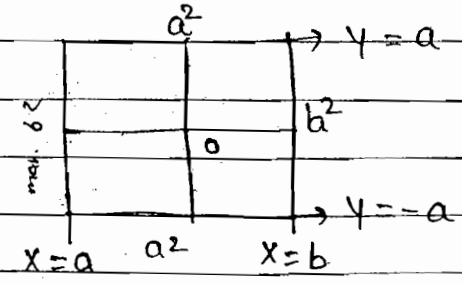
* Let u be a harmonic fun^c in a Simply domain D & conti. on boundary of D but still the max. of u is attained on the boundary of D but not inside.

Exp.

$$a, b > 0$$

$$u(x, y) = x^2 - y^2$$

$D =$



Let $a > b$:→ $b^2 < a^2$ $\quad \max\{a^2, b^2\}$

$$u(x, y) = b^2 - y^2$$

y is 0 then b^2

$$u(x, y) \quad x^2 - a^2 = b^2 - a^2 \le b^2$$

Ans b^2

Exp. $u = \sin x \cosh y$, $\frac{x^2}{2} + \frac{y^2}{3} = 1$

$u = \sin x \cosh \sqrt{3\left(1 - \frac{x^2}{2}\right)}$ $\quad y^2 = 3\left(1 - \frac{x^2}{2}\right)$

$\frac{\partial u}{\partial x} = \sin x \frac{e^t + \bar{e}^t}{2}$

* Minimum modulus principal :→ If f is non-vanishing analytic fun^c defined in D then mini $|f|$ is attained in D unless constant.

Minimum modulus Theorem :→ f is non-vanishing analytic on D & conti. on boundary of D then mini. $|f|$ is attained on ∂D but never attains in D.

⇒ f is analytic in D
conti on D
for any $z \in \partial D$ $\exists$ $z' \in D$
$|f'(z)| \leq |f(z)|$
Then either f is constant or $\exists$ $c \in D$
s.t. $\boxed{f(c) = 0}$
& $f(0) = 1/2$, $f(1/2) = 1/4$

Exp. $f(z) = z^2 + 2$, $|z| \leq 1 = D \cup \partial D$
$f(i) = 1$
mini $|f(z)| = 1$ ⟹
Mini on boundary.

Schwarz Lemma :→ $f : |z|<1$ analytic → D

s.t. (a) $f(0) = 0$ of order n

(b) $|f(z)| \le 1$

then —

(i) $|f(z)| \le |z|^n$

(ii) $|f^n(0)| \le \lfloor n$

(iii) If equality in (i) holds for some $c \in |z|<1$ then $f(z) \in z^n$

Sol^n:- (i) $f(z) = z^n \cdot g(z)$

$g(0) = a_n = \dfrac{f^n(0)}{n!}$

Now, $|z| < r < 1$

$$g(z) = \begin{cases} \dfrac{f(z)}{z^n}, & z \neq 0 \\ a_n = \dfrac{f^n(0)}{\lfloor n}, & z = 0 \end{cases}$$

g is analytic in $|z| < r$

$$\max |g(z)| = \max \left|\frac{f(z)}{z^n}\right| \le \frac{1}{z^n} = \frac{1}{r^n}$$

$$\max |g(z)| = \max \left|\frac{f(z)}{z^n}\right| \le 1$$

(i) $|f(z)| \le |z|^n$

(ii) $\max |g(z)| \le 1$

$\left|\dfrac{f(z)}{z^n}\right|$

$|f(z)| \le |z|^n$

$|g(0)| \le 1$

$\left|\dfrac{f^n(0)}{n!}\right| \le 1 \Rightarrow |f^n(0)| \le \lfloor n$

ii) Now $\exists\ c \in |z| < 1$

s.t. $|f(c)| \leq |c|^n$

$|g(c)| = 1$

$\max |g(z)| = |z(c)|$

g is analytic in $|z| < r$

$g(z) = \varepsilon$

$\frac{f(z)}{z^n} = \varepsilon \Rightarrow \boxed{f(z) = \varepsilon z^n}$

Thm :- $\phi : |z| < 1 \longrightarrow |z| < 1$

$\phi(z) = \left(\frac{a - z}{1 - \bar{a}z}\right) e^{i\lambda}, \quad |a| < 1$

A fun^c which is only bianalytic fun^c on unit disk i.e. fun^c which is bianalytic on unit disk has to be this form.

Now $\phi_\alpha = \frac{\alpha - z}{1 - \bar{\alpha}z} \quad |\alpha| < 1$

$\phi_\alpha(0) = \alpha, \quad \phi_\alpha(\alpha) = 0$

$\Rightarrow |\phi_\alpha(z)| \leq 1$

$(\phi_\alpha \circ \phi_\alpha)(z) = \phi_\alpha(\phi_\alpha(z))$

$= \frac{\alpha - \phi_\alpha(z)}{1 - \bar{\alpha}\phi_\alpha(z)}$

$(\phi_\alpha \circ \phi_\alpha)(z) = \frac{\alpha - \left(\frac{\alpha - z}{1 - \bar{\alpha}z}\right)}{1 - \bar{\alpha}\left(\frac{\alpha - z}{1 - \bar{\alpha}z}\right)} = z$

Put $\alpha = 0$ then $(\phi_\alpha \circ \phi_\alpha)(z) = 0$

Put $z = \alpha$ then $(\phi_\alpha \circ \phi_\alpha)(z) = \alpha$ //.

* Now $\phi_\alpha = \frac{\alpha - z}{1 - \bar{\alpha} z}$ $\quad |\alpha| < 1$

1) $\phi_\alpha(0) = \alpha$
2) $\phi_\alpha(\alpha) = 0$
3) $|\phi_\alpha(z)| \leq 1$
4) $\phi_\alpha^{-1} = \phi_\alpha$

(A) $f(a) = 0 \quad , \quad a \neq 0$
(B) $f(0) = a \quad , \quad a \neq 0$

$g_1(z) = (\phi_a \circ f)(z) = \phi_a(f(z)) \Rightarrow g(0) = 0$ when $f(0) = a$

$g_2(z) = (f \circ \phi_a)(z) = f(\phi_a(z)) = g_2(0) = 0$ when $f(a) = 0$

Some important inequation :→

1) If f satisfies Schwarz conditions then –

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}$$

ii) If $f(a) = 0$ for $a \in (0,1)$ and $f(a-1) = b \in (0,1)$ then $|b| \leq \frac{1}{1 + a - a^2}$

$f(1/3) = 0 \qquad f(-2/3) = ?$

$f(\frac{1}{3} - 1) = b$

$|b| = \frac{1}{1 + |1/3 - 1|9} = \frac{9}{11}$

$f(-\frac{2}{3}) = \frac{10}{11}$ Ans.

(iii) $f(z) \leq \frac{|\alpha - z|}{|1 - \bar{\alpha} z|}$ when $f(\alpha) = 0$

Q. $f(1/2) = 0$

$|f(3/4)| = ?$

$|f(z)| \leq \frac{\left|\frac{1}{2} - \frac{3}{4}\right|}{|1 - 3/8|} \leq \frac{2}{5}$ $\left(\begin{array}{l} \alpha = 3/4 \\ \alpha = 1/2 \end{array} \right)$

(iv) If $f(a) = 0$ &

$f(-a) = b$

s.t. $a \in (0,1)$, $b \in (0,1]$

where a, b are real

then $|b| \leq \frac{2a}{1+a^2}$

Q. $f(1/3) = 0$

$f(-1/3) = 4/5$

$\frac{2/3}{1 + 1/9} = \frac{3}{5}$

Residues :-

1) In expansion, coff. of $\frac{1}{z-a}$ at point 'a'

2) If fun^c has pole of order m then

$$= \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right]$$

3) If fun^c has simple pole then -

$$= \lim_{z \to a} \left[(z-a) f(z) \right]$$

4) If any of the real & imag. part of f is bdd then $f(z)$ is constant

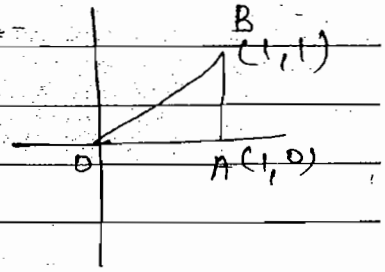
$w = f(z) = u + iv$

along γ, $w(t) = u(t) + i\,v(t)$

then $\int_\gamma f(z)\,dz = \int_a^b w(t)\,dt$

$= \int_a^b u\,t\,dt + i\int_a^b v(t)\,dt$

$\operatorname{Re}\int_\gamma f(z)\,dz = \int_a^b u(t)\,dt = \int_a^b \operatorname{Re}(w(t))\,dt$



Exp: $f(z) = (x^2+y^2) + (2x+1)i$

$u = x^2+y^2, \quad v = 2x+1$

find ?

$\gamma_1 = OA + OB$

$\gamma_2 = OB$

$\int_{\gamma_1} f(z)\,dz = \int_{OA} + \int_{AB}$

$= \left[\int_0^1 (x^2dx - 0) + i\int_0^1 (0+2x+1)\,dx\right] +$

$\left[\int_0^1 [0 - 3dy] + i\int_0^1 (1+y^2)\,dy + 0\right]$

$= \left(\frac{x^3}{3}\right)_0^1 + i\,(x^2+x)_0^1 + ((-3)\,y)_0^1 + i\left(y + \frac{y^3}{3}\right)_0^1$

$= -\frac{8}{3} + i\,\frac{10}{3}$

$\int_{\gamma_2} f(z)\,dz = \int_0^1 [2x^2dx - (2x+1)\,dx] + i\int_0^1 [2x^2 + (2x+1)\,dx$

$= -\frac{4}{3} + \frac{8}{3}i$

Q. $(x, y, z) = 3x^2 + 4y^2 - 6z^2 + 2yz - xz + xy$

$\left(0, 0, \sqrt{\frac{\alpha}{c}}\right) = -\alpha$, $\left(\sqrt{\frac{\alpha}{3}}, 0, 0\right) = \alpha$

Range $= \mathbb{R}$ $(-\infty, \infty)$

Imp 2014.

Q

$$ch_A(x) = x^3 - 2x^2 + 3x + 1$$

$$ch_{adj\,A}(x) = ?$$

$$ch_A(x) = a_0 + a_1 x + \cdots a_{n-1} x^{n-1} + x^n$$

$$adj\,A = \frac{(-1)^n a_0}{\lambda_1}, \frac{(-1)^n a_0}{\lambda_2} \cdots$$

$T \cup (x_1, x_2)$

$T(x_1, 0) = (0, x_1)$

$\cup (x_2, x_1) = (x_2, 0)$ X.

$(x_2, x_1) = (x_1, x_2)$ X.

$T(x_1, x$ $T(x_1, x$

$(x_1, 0) = (x_1$